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INCREMENTAL PLASTIC ANALYSIS IN THE PRESENCE OF LARGE DISPLACEMENTS AND PHYSICAL INSTABILIZING EFFECTS†

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Abstract—Elastic–plastic constitutive laws are assumed with allowance for the following: lack of normality, corners with interaction between yielding modes and work-softening. Continua are replaced by finite element models which fulfill compatibility throughout. Equilibrium equations refer to the deformed configuration; strains, however, are presumed small. The structural response to rates of loads and dislocations (e.g. thermal strains) is studied and the following results are obtained; (a) six extremum properties of the solutions, which reduce the incremental problem to a quadratic programming problem; two of these properties are valid without restrictions; (b) methods for obtaining lower and upper bounds to the local instantaneous compliance with respect to a single load or dislocation; (c) criteria for the uniqueness of solution to the incremental problem and for the overall stability of the system.

1. INTRODUCTION

THE plastic analysis of structures undergoing large deformations represents on the one hand a particularly difficult, still unsettled field of nonlinear mechanics, on the other hand an almost mandatory task for the engineer in many practical stiuations. It is not surprising therefore, that an intensive research effort has been and is being devoted to the many relevant problems. In the abundant literature available it seems possible to distinguish results mainly useful for the theoretical framework of the fenomenology in question and a stream of studies predominantly aimed at the numerical analysis of specific categories of structures. On the theoretical side at least the traditional references to the contributions of Shanley [1] and Hill [2, 3] must be given as a first orientation. On the application-oriented side, a survey of the achievements up to 1959 can be found in Horne [4].

The latter stream of research in the field of large deformations has been recently joined by the tremendous development of the matrix methods of structural analysis. Finite element discretization has proved extremely suitable for studying continua of any shape, even when nonlinearity either of geometrical or of physical nature or both are present. However, finite element models do not only supply a basis for approximate numerical solutions of complicated problems, but also can represent an alternative convenient description of the behaviour of mechanical systems.

As emphasized by Besseling [5, 6], also the traditional continuous field description gives but an average, in the neighborhood of each point, of phenomena which are inherently discontinuous on a microscopic scale. The vector-matrix representation in terms of lumped field quantities, if these are properly chosen, is less detailed (as detailed as one

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actually needs), but by no means less general: all properties and features of the mechanical behaviour of solid bodies are equally reflected both in the continuous and in the discrete description. The differences consist merely in formal aspects, and the transfer of physical notions from one to another representation is an easy matter. Therefore it is clearly possible to develop a theory of the structural behaviour founded on the latter description.

Some arguments in favour of such a theory are quite general: the mathematical tools of matrix algebra appear more manageable and open to the physical insight than those of the functional analysis; moreover they are ideally suited to be directly programmed for computers.

In plasticity, matrix theories appear to be particularly advantageous, since plastic systems are governed by basic relation sets which involve both equations and inequalities. Properties and solution procedures of certain sets of algebraic equations and inequalities and their connexions with optimization problems have been recently well explored in some modern fields of applied mathematics (operations research, programming theory, theory of games). We have pointed out in previous works that several mathematical results in these fields apply to the relation sets which govern discrete plastic systems, and meet well the requirements of both their theory and engineering analysis [7, 8]. Continuous plastic systems give rise to sets of (differential) equations and inequalities, to be discussed in a mathematical framework which is not only relatively more sophisticated but also, to the author's knowledge, less developed and consolidated at present, despite outstanding achievements due to Moreau *et al.* (see e.g. [9]).

The above considerations suggest as worthwhile and fruitful the combination of finite element discretization and mathematical results from programming theory and related topics, both in order to implement the knowledge of the structural plastic behaviour in the presence of instabilizing effects, and to introduce new techniques for the numerical solution of practical engineering problems.

The present paper is intended as a contribution in both these directions.

First (Section 2) we replace continua of any dimensions and shape by finite element models, which imply homogeneous stress and strain fields in each element and comply with compatibility everywhere, but whose node and mesh pattern does not need to be specified. "Natural" generalized stresses and strains are adopted to define the statical regime of each element and this concept proves once again productive of clearness and compactness for the subsequent analysis.

The compatibility and equilibrium equations for the finite element assembly can be linearized for the "incremental" processes provoked by infinitesimal external actions added to a current known situation. Only such incremental problems are considered in the paper, being understood that any load and/or dislocation (e.g. thermal) history may be followed as a sequence of incremental processes of properly chosen small amplitude. For a detailed discussion of these preliminaries the reader is referred to some leading authors in matrix methods of mechanics such as [10, 11]. Displacements are usually called "large", when they affect the equilibrium equations. Strains are not necessarily large as a consequence; we shall suppose that they are not (as it happens to be in most structural problems), so that the constitutive laws can be formulated as in the small displacement theory, provided that they be expressed in intrinsic (unaffected by rigid body motions) stress and strains, as the "natural" components actually are.

The constitutive laws adopted herein (directly for the finite elements), are more general than the traditional ones. In singular points of the yield surfaces, "interaction" between yielding modes is admitted, according to Mandel's generalization of Koiter's theory [12]. Moreover. Drucker's stability postulate [13] may be violated for either lack of normality (of the plastic strain vector to the yield locus) or work-softening or both. These features of the material behaviour can be called "physical instabilizing effects": in fact the subsequent analysis will show that, according to the spirit of Drucker's postulate, they play a role comparable to that of the instabilizing effects due to large displacements (which shall be referred to here as "geometrical instabilizing effects", although, clearly, they may act in favour of stability as well). The assumed general, "linear" kind of flow rules have been examined in [8]. The external actions may include both (nodal) forces, supposed conservative (dead loading) and imposed strains or displacements (dislocations e.g. of thermal nature).

In Section 3 three extremun principles are established which reduce the incremental problem in question to optimizations of quadratic functionals under linear equations and inequality constraints, assuming as variables the displacement rates and the plastic multiplier rates. The first principle is characterized by the interesting and, in a sense, striking property of unrestricted validity throughout the field where the basic weak assumptions hold. In contrast to this "general" theorem, the second and third characterizations of solutions can be proved only under the conditions of normality, reciprocity interaction between yielding modes and convexity of a certain quadratic form (the last condition represents a limitation on the cumulative instabilizing effect of large displacements and softening). However these "particular" properties hold in a range broader than Koiter's classical potential theory. From them we derive by specialization some already known results [14, 16, 17], among which Hill's minimum principle [2], originally established, however, with allowance for large strains too.

The ease and compactness of all proofs are due to the fact that deliberate systematic use is made of the aforementioned mathematical results, the essence of which is outlined in the Appendix for the reader's convenience. These results offer the additional advantage of emphasizing the intimate structure of the mechanical theory developed: thus the extremum properties (II) and (III) correspond to a pair of "dual symmetric" quadratic programs, of which property (I) is the "selfdual, composite" program.

In Section 4 three extremum principles for the plastic multiplier rates are obtained. The levels of generality and the mathematical structure are the same as for the developments of Section 3. The new statements are shown to cover as special cases some already known results [18–20].

In Section 5 the former pair of dual extremum theorems is used in order to bound both from below and above (without actually solving the incremental problem) the instantaneous stiffness that the system exhibits locally with respect to a single load or dislocation component.

In the presence of instabilizing effects, uniqueness and existence of solutions are important questions, closely connected with the stability analysis. These topics are discussed in Sections 6 and 7, and the relevant analytical criteria are established. Besides new results and generalizations, also known notions flow straightforwardly, such as Shanley's concept of stable equilibrium bifurcation [1], Hill's sufficient criteria for uniqueness and stability in the presence of geometrical second order effects only [2], and the uniqueness and stability criteria previously given in Ref. [21] for the cases where only physical instabilizing effects are present.

The transfer of results from the vector-matrix theory to the tensorial field description may not be obvious; therefore this point is discussed in Section 8.

The theoretical treatment developed seems to have an operative value, as it leads directly to fairly efficient numerical techniques, for solving the incremental problem, bounding the local instantaneous compliances and for the practical checking of stability or uniqueness. In Section 9 the relevant algorithms are indicated by referring the reader to the specialized literature for detailed information.

The extension of the theory to allow for large strains and its adjustment to the analysis of rigid–plastic systems will be discussed elsewhere.

2. BASIC RELATIONS

2.1 *Finite element discretization and notation*

The matrix-vector description of structural behaviour used in this study, rests on the following notions, which have become customary in finite element analysis of continua [10, 11, 22].

(a) Three- and two-dimensional continua are represented by an assembly of tetrahedral and triangular finite elements respectively, by selecting a suitable set of vertices ("nodes").

(b) Within each element the variation of displacements is prescribed to be linear and, therefore, the stress and strain fields constant (homogeneous). Thus compatibility is fulfilled everywhere, equilibrium only at the nodal points, where all external forces are assumed acting.

(c) Each element is assigned average homogeneous properties, which may vary from one to another in order to allow for dishomogeneity.

(d) The elemental stress and strain states are defined by means of "intrinsic" or "*natural*" generalized stress and strain components respectively.†

(e) Displacements possibly imposed on the constrained boundary of the body are interpreted as dislocations prescribed within additional rigid bars or elements of suitable location and direction.

In what follows, we indicate column-vectors and matrices by bold face letters; a superposed tilde means transpose, a superposed dot derivative with respect to time (rate). Vector inequalities apply to each pair of corresponding components; **0** denotes vectors or matrices whose entries are all zero.

On the basis of the above discretization, the current configuration of the continuum considered will be described by the vector **u** of nodal displacements (with reference to an "initial" configuration and to fixed Cartesian axes common for the whole system), the current external load distribution by the analogous vector **F**; the current states of stresses, strains and imposed dislocations thoughout the body will be defined by the vectors

$$\tilde{\boldsymbol{\sigma}} \equiv [\tilde{\boldsymbol{\sigma}}^1 \,\vdots\, \dots \,\vdots\, \tilde{\boldsymbol{\sigma}}^n]; \qquad \tilde{\boldsymbol{\varepsilon}} \equiv [\tilde{\boldsymbol{\varepsilon}}^1 \,\vdots\, \dots \,\vdots\, \tilde{\boldsymbol{\varepsilon}}^n]; \qquad \tilde{\boldsymbol{\delta}} \equiv [\tilde{\boldsymbol{\delta}}^1 \,\vdots\, \dots \,\vdots\, \tilde{\boldsymbol{\delta}}^n] \tag{2.1}$$

† The natural variables (a concept introduced and discussed by Argyris *et al.* [10, 22]) are proportional to direct stresses and strains measured parallel to the element edges and represent nodal forces along the edges and edge elongations respectively (6 components for tetrahedra, 3 for triangles, 1 for pin-ended bars). They might be called also "independent" or "intrinsic" in view of the fact that the generalized strains so defined are unaffected by rigid body motions, the corresponding generalized stresses are selfequilibrated.

formed by assembling as subvectors, in a fixed order, all the generalized variables vectors $\boldsymbol{\sigma}^i$, $\boldsymbol{\varepsilon}^i$, $\boldsymbol{\delta}^i$ which govern the corresponding fields within each of the n elements of the structure.

2.2 Constitutive relations

The behaviour of any element, described by the $\boldsymbol{\varepsilon}^i(\boldsymbol{\sigma}^i)$ relationship, reflects all features of the behaviour of its material described by the $\varepsilon^i_{rs}(\sigma^i_{rs})$ relation, because of the one-to-one linear and contragredient character of the correspondences between natural generalized variable vectors and element fields (for explicit formulations of these correspondences see e.g. [10, 22]). Therefore we may introduce the constitutive laws by referring directly to elements instead of to materials. The incremental elastoplastic stress–strain relations adopted herein will be of the general "*linear*" type we discussed in [8].

These rate relations can be analytically represented as follows (for the generic ith finite element):

$$\dot{\boldsymbol{\varepsilon}}^i = \dot{\boldsymbol{\varepsilon}}^{ie} + \dot{\boldsymbol{\varepsilon}}^{ip} + \dot{\boldsymbol{\delta}}^i \tag{2.2}$$

$$\dot{\boldsymbol{\varepsilon}}^{ie} = (\mathbf{S}^i)^{-1}\dot{\boldsymbol{\sigma}}^i \tag{2.3}$$

$$\dot{\boldsymbol{\varepsilon}}^{ip} = \mathbf{V}^i\dot{\boldsymbol{\lambda}}^i \tag{2.4}$$

$$\dot{\boldsymbol{\varphi}}^i = \tilde{\mathbf{N}}^i\dot{\boldsymbol{\sigma}}^i - \mathbf{H}^i\dot{\boldsymbol{\lambda}}^i \tag{2.5}$$

$$\dot{\boldsymbol{\lambda}}^i \geq \mathbf{0},\ \dot{\boldsymbol{\varphi}}^i \leq \mathbf{0} \tag{2.6}$$

$$\tilde{\dot{\boldsymbol{\varphi}}}^i\dot{\boldsymbol{\lambda}}^i = 0 \tag{2.7}$$

Equation (2.2) distinguishes the elastic, plastic and dislocation (e.g. thermal) contributions to the strain rates. $\mathbf{S}^i$ is the ("natural", symmetric, positive definite) elastic stiffness matrix of the ith element. Matrix $\mathbf{N}^i \equiv [\mathbf{N}^i_1 \ldots \mathbf{N}^i_{s^i}]$ collects as columns the gradients $\mathbf{N}^i_j \equiv (\partial\varphi^i_j/\partial\boldsymbol{\sigma}^i)_{\boldsymbol{\sigma}^{Yi}}$ of the s^i *yield functions* φ^i_j which are zero in the current stress state $\boldsymbol{\sigma}^{Yi}$ and, hence, represent yield modes which can be activated in the incremental process. $\mathbf{V}^i \equiv [\mathbf{V}^i_1 \ldots \mathbf{V}^1_{s^i}]$ is the analogous matrix formed by the gradients $\mathbf{V}^i_j \equiv (\partial\psi^i_j/\partial\boldsymbol{\sigma}^i)_{\boldsymbol{\sigma}^{Yi}}$ of the corresponding *plastic potentials* ψ^i_j; vectors $\tilde{\dot{\boldsymbol{\varphi}}}^i \equiv [\dot{\varphi}^i_1 \ldots \dot{\varphi}^i_{s^i}]$ and $\tilde{\dot{\boldsymbol{\lambda}}}^i \equiv [\dot{\lambda}^i_1 \ldots \dot{\lambda}^i_{s^i}]$ include the yield function rates and the plastic multiplier rates of the s^i activable modes. $\mathbf{H}^i$ denotes the $s^i \times s^i$ *interaction work-hardening matrix*. For a detailed discussion of the above flow laws see [8]. Their meaning and generality become clearer by referring to Fig. 1 (where $s^i = 2$) and by noting the following special cases.

The stress states from which the incremental process starts, is either *elastic* or corresponds to a *regular* or to a *singular* point of the instantaneous yield surface, depending on whether $s^i = 0$, $s^i = 1$, $s^i > 1$, respectively. The circumstance $\mathbf{N}^i = \mathbf{V}^i$ reflects *normality*: then $\mathbf{H}^i = \mathbf{0}$ characterizes perfectly plastic behaviour, $\mathbf{H}^i \equiv \text{diag}[H^i_1 \ldots H^i_{s^i}]$ with $H^i_j > 0$ implies workhardening according to Koiter's hypothesis [14]; a nondiagonal but symmetric, positive semidefinite $\mathbf{H}^i$ allows for *interaction* between plastic modes as in Mandel's generalization [12] of Koiter's theory.

We do not *a priori* assume any restrictive hypothesis on s^i, $\mathbf{N}^i$, $\mathbf{V}^i$, $\mathbf{H}^i$ (which are, moreover, history-dependent entities) except the weak hypothesis that the vectors of each set $\mathbf{N}^i_j$ and $\mathbf{V}^i_j$ ($j = 1 \ldots s^i$) are linearly independent and all scalar products $\tilde{\mathbf{N}}^i_j\mathbf{V}^i_j$ are positive. Therefore some non traditional but theoretically and technically important features of plastic phenomenology can be covered by the present theory, namely:

lack of normality (*nonassociated flow-laws*) $\mathbf{V}^i \neq \mathbf{N}^i$; *softening* (local regression of the yield surface by yielding), $H^i_{hh} < 0$; *lack of reciprocity* in the interaction effects, $H^i_{hk} \neq H^i_{kh}$. *Anisotropy* is excluded in neither the elastic nor the plastic ranges, since the restrictions on $\mathbf{S}^i$ and φ^i_j required by isotropy are nowhere needed in what follows.

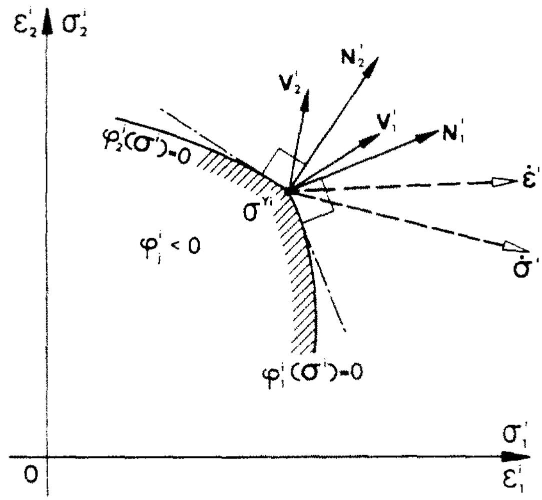

FIG.1. A graphical interpretation of the incremental stress–strain laws in two dimensions.

It is convenient for future developments to write in a compact form the relation between the vectors $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$, equation (2.1), which depict the desired incremental stresses and strains throughout the structure subdivided in n finite elements. To this end let us form the super-vectors $\bar{\boldsymbol{\lambda}} \equiv [\bar{\boldsymbol{\lambda}}^1 \ldots \bar{\boldsymbol{\lambda}}^m]$, $\bar{\boldsymbol{\phi}} \equiv [\bar{\boldsymbol{\phi}}^1 \ldots \bar{\boldsymbol{\phi}}^m]$ and the diagonal block matrices $\mathbf{S} \equiv \text{diag}[\mathbf{S}^1 \ldots \mathbf{S}^m]$, $\mathbf{H} \equiv \text{diag}[\mathbf{H}^1 \ldots \mathbf{H}^m]$, $\mathbf{V}^d \equiv \text{diag}[\mathbf{V}^1 \ldots \mathbf{V}^m]$, $\mathbf{N}^d \equiv \text{diag}[\mathbf{N}^1 \ldots \mathbf{N}^m]$, where m is the number of the elements at the yield point. Moreover, let matrix $\mathbf{V} \equiv \begin{bmatrix} \mathbf{V}^d \\ \hline \mathbf{0} \end{bmatrix}$ be such that the plastic strain rate vector $\dot{\boldsymbol{\varepsilon}}^P$ for the whole structure can be expressed as $\mathbf{V}\dot{\boldsymbol{\lambda}}$; analogously, let $\mathbf{N} \equiv \begin{bmatrix} \mathbf{N}^d \\ \hline \mathbf{0} \end{bmatrix}$. By means of these new symbols, the constitutive laws (2.2)–(2.7) can be assembled, for $i = 1 \ldots n$, in the relation set:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} + \mathbf{V}\dot{\bar{\boldsymbol{\lambda}}} + \dot{\boldsymbol{\delta}} \tag{2.8}$$

$$\dot{\boldsymbol{\phi}} = \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}} - \mathbf{H}\dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} \geq \mathbf{0}, \qquad \dot{\boldsymbol{\phi}} \leq \mathbf{0}, \qquad \tilde{\dot{\boldsymbol{\phi}}}\dot{\boldsymbol{\lambda}} = 0. \tag{2.9a, b, c, d}$$

The dependence $\dot{\boldsymbol{\lambda}}(\dot{\boldsymbol{\sigma}})$, which governs the plastic incremental response of the element assembly, is expressed by (2.9) in the form of a "linear complementary problem" defined by matrix $\mathbf{H}$ and vector $\tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}$ (cf. [8] and Appendix A).

As they concern intrinsic quantities under the hypothesis of small strains, the above relations are not affected by the presence of large displacements.

2.3 *Compatibility equations*

An infinitesimal geometrical perturbation, i.e. a nodal velocity vector $\dot{\mathbf{u}}$, superposed to a given configuration Y, uniquely defines a strain rate distribution through the linear transformation

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{B}\dot{\mathbf{u}}. \tag{2.10}$$

In regimes of large displacements, *the compatibility matrix* $\mathbf{B}$ depends on the configuration $Y(\mathbf{u})$, but not on the configuration changes $\dot{\mathbf{u}}$.†

2.4 *Equilibrium equations*

In a quasi-static regime of large displacements equilibrium relates external load rates to natural stress rates (in the sense specified above) through the equation:

$$\tilde{\mathbf{B}}\dot{\boldsymbol{\sigma}} + \mathbf{K}_G\dot{\mathbf{u}} = \dot{\mathbf{F}} \tag{2.11}$$

where $\mathbf{K}_G$ represents the *geometric stiffness matrix* of the assembly: $\mathbf{K}_G$ is symmetric and depends on the geometry and, linearly, on the stress state $\boldsymbol{\sigma}^Y$ pertaining to the situation Y from which the infinitesimal change considered starts; it is not dependent on the increments of the variables and on the material properties. For a derivation and detailed discussion of equation (2.11) and in particular of $\mathbf{K}_G$ we refer the reader to Argyris, [10, 22].

The set of relations (2.8)–(2.11) completely governs the static behaviour of the system in the neighborhood of a given equilibrated configuration Y. In contrast to the small displacement analysis, the compatibility matrix varies with the considered situation Y and the equilibrium equations acquire the corrective term $\mathbf{K}_G\dot{\mathbf{u}}$: in the present formulation, the substantial novelty of the incremental analysis problem for large displacements concentrates in this term only.

3. EXTREMUM PRINCIPLES FOR DISPLACEMENT AND PLASTIC MULTIPLIER RATES

3.1 *The* $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$ *formulation*

The governing relation set (2.8)–(2.11) contains five unknown variable vectors: $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\sigma}}$, $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$. By substituting (2.8) into (2.10) and solving the latter equation with respect to $\dot{\boldsymbol{\sigma}}$‡ we obtain:

$$\dot{\boldsymbol{\sigma}} = \mathbf{S}\mathbf{B}\dot{\mathbf{u}} - \mathbf{S}\mathbf{V}\dot{\boldsymbol{\lambda}} - \mathbf{S}\dot{\boldsymbol{\delta}} \tag{3.1}$$

Substitutions of (3.1) into the equilibrium equation (2.11) and into (2.9a), lead to the following alternative set of governing relations in only three unknown vectors:

$$(\tilde{\mathbf{B}}\mathbf{S}\mathbf{B} + \mathbf{K}_G)\dot{\mathbf{u}} - \tilde{\mathbf{B}}\mathbf{S}\mathbf{V}\dot{\boldsymbol{\lambda}} = \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{3.2}$$

$$\dot{\boldsymbol{\varphi}} = \tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\dot{\mathbf{u}} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V})\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{3.3}$$

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0}, \qquad \dot{\boldsymbol{\varphi}} \leq \mathbf{0}, \qquad \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0. \tag{3.4a, b, c}$$

Let us put:

$$\mathbf{K} = \tilde{\mathbf{B}}\mathbf{S}\mathbf{B} + \mathbf{K}_G = \mathbf{K}_E + \mathbf{K}_G \tag{3.5}$$

observing that for $\mathbf{K}$ the denomination of instantaneous (external) *elastic stiffness* matrix is appropriate: in fact, should any further yielding be precluded, the displacement response to external action increments added to the situation Y, would be simply given by

$$\mathbf{K}\dot{\mathbf{u}} = \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}}. \tag{3.6}$$

† $\mathbf{B}$ can be expressed as the product of a matrix which depends only on the current geometry of the system (direction cosines of the element edges at Y) and of a matrix whose entries are 0 and 1 ("location" or "Boolean" matrix [10, 22]).

‡ The (internal) stiffness matrix $\mathbf{S}$ of the assembled structure cannot be singular, since $\mathbf{S}$ is a diagonal supermatrix, each ith entry of which is a submatrix given by the element *natural* stiffness matrix $\mathbf{S}^i$ which appears in (2.2).

It is worth stressing that $\mathbf{K}$ may be singular and/or nondefinite, as it consists of the *purely elastic* $\mathbf{K}_E$ and the *geometric* $\mathbf{K}_G$ addends: the former positive definite † addend $\mathbf{K}_E$ would represent the elastic stiffness of the assembly only if there were no second order geometrical effects.

3.2 *A "general" minimum theorem for displacement and plastic multiplier rates*

It is convenient to introduce the new symbols

$$\mathbf{C}_V \equiv \tilde{\mathbf{B}}\mathbf{S}\mathbf{V}, \qquad \mathbf{C}_N \equiv \tilde{\mathbf{B}}\mathbf{S}\mathbf{N} \tag{3.7a, b}$$

$$\mathbf{G} \equiv \mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V} \tag{3.8}$$

and to rewrite the relation set (3.2), (3.3), (3.4b) in the form:

$$\begin{aligned} \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} - \mathbf{K}\dot{\mathbf{u}} + \mathbf{C}_V\dot{\boldsymbol{\lambda}} &= \dot{\mathbf{v}}^- \geq \mathbf{0} \\ -(\dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}}) + \mathbf{K}\dot{\mathbf{u}} - \mathbf{C}_V\dot{\boldsymbol{\lambda}} &= \dot{\mathbf{v}}^+ \geq \mathbf{0} \end{aligned} \tag{3.9}$$

$$\tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\delta}} - \tilde{\mathbf{C}}_V\dot{\mathbf{u}} + \mathbf{G}\dot{\boldsymbol{\lambda}} = -\dot{\boldsymbol{\varphi}} \geq \mathbf{0} \tag{3.10}$$

where $\dot{\mathbf{v}}^-$, $\dot{\mathbf{v}}^+$ are vectors of auxiliary "slack" variables, which are required to vanish by the equations and inequalities (3.9) which involve them. Let us express the displacement vector as:

$$\dot{\mathbf{u}} = -\dot{\mathbf{u}}^- + \dot{\mathbf{u}}^+ \quad (3.10), \quad \text{where} \quad \dot{\mathbf{u}}^- \geq \mathbf{0},\ \dot{\mathbf{u}}^+ \geq \mathbf{0} \tag{3.11}$$

and substitute (3.10) into (3.9).

By assuming:

$$\mathbf{q} \equiv \begin{bmatrix} \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \\ \hdashline -\dot{\mathbf{F}} - \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \\ \hdashline \tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\delta}} \end{bmatrix}; \qquad \boldsymbol{\zeta} \equiv \begin{bmatrix} \dot{\mathbf{u}}^- \\ \hdashline \dot{\mathbf{u}}^+ \\ \hdashline \dot{\boldsymbol{\lambda}} \end{bmatrix}; \qquad \boldsymbol{\omega} \equiv \begin{bmatrix} \dot{\mathbf{v}}^- \\ \hdashline \dot{\mathbf{v}}^+ \\ \hdashline \dot{\boldsymbol{\varphi}} \end{bmatrix} \tag{3.12a, b, c}$$

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{K} & -\mathbf{K} & \mathbf{C}_V \\ -\mathbf{K} & \mathbf{K} & -\mathbf{C}_V \\ \tilde{\mathbf{C}}_V & -\tilde{\mathbf{C}}_V & \mathbf{G} \end{bmatrix} \tag{3.13}$$

we notice that (3.9), i.e. (3.2), (3.3), (3.4b), can be written as:

$$\mathbf{q} + \mathbf{M}\boldsymbol{\zeta} = \boldsymbol{\omega}; \qquad \boldsymbol{\omega} \geq \mathbf{0} \tag{3.14a, b}$$

whereas the remaining (3.4b, c) (3.11) can be expressed by:

$$\boldsymbol{\zeta} \geq \mathbf{0}; \qquad \tilde{\boldsymbol{\omega}}\boldsymbol{\zeta} = 0. \tag{3.15a, b}$$

Thus we have reduced the $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$ formulation of the mechanical problem in hand to the linear *complementarity problem* (3.14), (3.15). This in turn is equivalent to the *quadratic programming* problem:

$$\text{minimize} \qquad Q \equiv \tilde{\boldsymbol{\zeta}}\mathbf{M}\boldsymbol{\zeta} + \tilde{\mathbf{q}}\boldsymbol{\zeta} \tag{3.16}$$

† $\mathbf{B}$ is full column-rank (i.e. has linearly independent columns) by its very definition through equation (2.10) hence the definiteness of $\mathbf{S}$ implies that of $\mathbf{K}_E$.

subject to

$$\zeta \geq 0, \qquad \mathbf{M}\zeta + \mathbf{q} \geq \mathbf{0} \tag{3.17}$$

provided that the optimal value of this be zero (see Appendix A.1).

By substituting equations (3.12), (3.13) into (3.16), (3.17), performing some algebraic manipulations and, finally, making use of equations (3.5), (3.7), (3.8), we reach from (3.16), (3.17) the following conclusion:

(I) *In the class of all distributions of velocities* $\dot{\mathbf{u}}$ *and plastic multiplier rates* $\dot{\lambda}$, *which satisfy the linear constraints*

$$\dot{\lambda} \geq \mathbf{0} \tag{3.18a}$$

$$\tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\dot{\mathbf{u}} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{V})\dot{\lambda} \leq \tilde{\mathbf{N}}\mathbf{S}\dot{\delta} \tag{3.18b}$$

$$(\mathbf{K}_E + \mathbf{K}_G)\dot{\mathbf{u}} - \tilde{\mathbf{B}}\mathbf{S}\mathbf{V}\dot{\lambda} = \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\delta} \tag{3.19}$$

the/a solution, if any, of the incremental problem for given load and dislocation rates $\dot{\mathbf{F}}, \dot{\delta}$, *minimizes the quadratic functional*

$$Q(\dot{\mathbf{u}}, \dot{\lambda}) \equiv \tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tilde{\dot{\lambda}}\mathbf{H}\dot{\lambda} + (\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \tilde{\dot{\lambda}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\dot{\mathbf{u}} - \mathbf{V}\dot{\lambda}) - \tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}} - \tilde{\dot{\delta}}\mathbf{S}(\mathbf{B}\dot{\mathbf{u}} - \mathbf{N}\dot{\lambda}). \tag{3.20}$$

The absolute maximum of Q, if zero, characterizes the solution; if not zero no solution exists.

The inequalities (3.18) come from the constitutive laws (the latter from the nonpositive requirement on the rates of the yield functions which are zero in Y); equation (3.19) expresses equilibrium, but also compatibility is implicitly taken into account in the elimination of $\dot{\varepsilon}$. Therefore the above statement essentially gives an extremal formulation of the nonlinear requirement contained in the plastic flow rules, where all nonlinearity of the incremental problem is included.

Theorem (I) is "general" in the sense that, to be valid, it requires no further restrictions besides those, very weak indeed, assumed in formulating the governing equations. It is "*selfdual*" because the dualization (in the sense of programming theory) of its mathematical expression would lead to the same conclusion (see Appendix A.2).

A most desirable special property is the positive semidefiniteness of matrix **M**, which makes the energy function convex and many wellknown quadratic programming algorithms applicable; this property, however, does not imply symmetry, and, hence, neither normality, nor reciprocity interaction.

3.3 "*Particular*" *extremum theorems for displacement and plastic multiplier rates*

3.3.1. Suppose now that matrix **M** be both *symmetric and positive semidefinite*, which means that the nonlinear part of $Q(\dot{\mathbf{u}}, \dot{\lambda})$ is nonnegative whatever $\dot{\mathbf{u}}, \dot{\lambda}$ may be [hypothesis (3.21)]. Through (3.5), (3.7), (3.8), (3.13), it appears that: **M** is symmetric if, and only if, normality holds ($\mathbf{V} = \mathbf{N}$) and the interaction between yielding modes exhibits reciprocity ($\tilde{\mathbf{H}} = \mathbf{H}$). It is easily seen from (3.20) that, when $\mathbf{V} = \mathbf{N}$, a *sufficient not necessary* condition for the positive semidefiniteness of **M** is the validity of the same property for **H** (nonsoftening) and $\mathbf{K}_G$ (geometrical effects not instabilizing); a much weaker, *necessary* not sufficient condition is the positive semidefiniteness of matrices **G** and **K**.

Let us now compare the relation set (3.14), (3.15) to (A3) of Appendix A.2. Through the identifications:

$$\mathbf{A} = \mathbf{0}, \qquad \mathbf{E} = \mathbf{0}, \qquad \mathbf{D} = \mathbf{M}, \qquad \mathbf{c} = \mathbf{q}, \qquad \mathbf{b} = \mathbf{0} \tag{3.22}$$

the set (3.14), (3.15) appears to be a special case of (A3), the above assumption on $\mathbf{M}$ coincides with the hypothesis (A5) and, therefore, as pointed out in A.2, the linear complementarity problem (3.4), (3.15) is fully equivalent to a convex optimization problem of kind (A.6), which, via (3.22), reads:

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\tilde{\boldsymbol{\zeta}}\mathbf{M}\boldsymbol{\zeta}+\tilde{\mathbf{q}}\boldsymbol{\zeta} \\ &\text{subject to} \quad \boldsymbol{\zeta} \geq \mathbf{0} \end{aligned} \tag{3.23}$$

Substituting into (3.23) equations (3.12), (3.13) and, subsequently, equations (3.5), (3.8), (3.7) with $\mathbf{V} = \mathbf{N}$, after a little algebra, the new mathematical model (3.23) of the mechanical problem in hand acquires the form specified in the following statement:†

(II) *Under the hypothesis* (3.21), *in the class of all distributions of displacement rates* $\dot{\mathbf{u}}$ *and plastic multiplier rates* $\dot{\boldsymbol{\lambda}}$ *which fulfill the sign restriction*

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0} \tag{3.24}$$

the/any solution, if any, of the incremental problem for given straining effect rates $\dot{\mathbf{F}}, \dot{\boldsymbol{\delta}}$, *is characterized by the maximum of the quadratic function*:

$$Q_1(\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}) \equiv -\tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}}-\tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}}-\tfrac{1}{2}(\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}}-\tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\dot{\mathbf{u}}-\mathbf{N}\dot{\boldsymbol{\lambda}})+\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}}+\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}(\mathbf{B}\dot{\mathbf{u}}-\mathbf{N}\dot{\boldsymbol{\lambda}})-\underline{\tfrac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}\dot{\boldsymbol{\delta}}}. \tag{3.25}$$

Note that the optimization postulated by Theorem (II) presumes compatibility and stands for explicit use of equilibrium and some flow rules ($\dot{\boldsymbol{\varphi}} \leq \mathbf{0}$ and $\tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0$).

By means of the compatibility and conformity requirements (2.10), (2.8) and (2.3), we obtain from (3.24), (3.25) the alternative form:

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}}+\tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}}+\tfrac{1}{2}\tilde{\dot{\boldsymbol{\varepsilon}}}^e\mathbf{S}\dot{\boldsymbol{\varepsilon}}^e-\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}} \tag{3.26}$$

$$\text{subject to} \quad \dot{\boldsymbol{\varepsilon}}^e = \mathbf{B}\dot{\mathbf{u}}-\mathbf{N}\dot{\boldsymbol{\lambda}}-\dot{\boldsymbol{\delta}}, \qquad \dot{\boldsymbol{\lambda}} \geq \mathbf{0}. \tag{3.27a, b}$$

A further consequence of Theorem (II) can be obtained, by observing that full conformity to the constitutive laws can be imposed in the optimization process without affecting its results, since any solution clearly complies with this requirement. From (2.9a, d) it follows that:

$$\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} = \tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}} = \tilde{\dot{\boldsymbol{\sigma}}}\dot{\boldsymbol{\varepsilon}}^p. \tag{3.28}$$

Substituting equations (3.28), (3.27a) and (2.10) into (3.26), under the condition that for any strain rate distribution $\dot{\boldsymbol{\varepsilon}}$ resulting from any generic $\dot{\mathbf{u}}$, the corresponding (through the flow-laws) stress rates $\dot{\boldsymbol{\sigma}}$ are taken,‡ we may write, instead of (3.26), (3.27):

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}}+\tfrac{1}{2}\tilde{\dot{\boldsymbol{\sigma}}}(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\delta}})-\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}} \tag{3.29}$$

$$\text{subject to} \quad \dot{\boldsymbol{\varepsilon}} = \mathbf{B}\dot{\mathbf{u}}. \tag{3.30}$$

The above conclusions cover as special cases of decreasing generality the following previous results: (i) for small displacements ($\mathbf{K}_G = \mathbf{0}$), and non interacting yielding modes ($\mathbf{H}$ diagonal), Theorem (II) and (3.29), (3.30) reduce to the two statements proved in [17]:

† For future convenience we transform the minimization to a maximization by changing the sign of the objective function, and add to the objective the immaterial constant $-\frac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}\dot{\boldsymbol{\delta}}$ (underlined to indicate that it can be dropped for the optimization).

‡ It has been pointed out in [8] that there exists a unique $\dot{\boldsymbol{\sigma}}^t$ for any $\dot{\boldsymbol{\varepsilon}}^t$ if, and only if, all principal minors of matrix $\mathbf{G}^t = \mathbf{H}^t - \tilde{\mathbf{N}}^t\mathbf{S}^t\mathbf{N}^t$ are positive.

(ii) in the absence of imposed (e.g. thermal) strainrates ($\dot{\boldsymbol{\delta}} = 0$) they become the theorems proved by Capurso [16] and Greenberg [13, 14] respectively: (iii) in the elastic range both properties coincide in the classical potential energy principle applied to incremental responses.

The theorem expressed by (3.29), (3.30) was established by Hill [2] with allowance for large strains, but for smooth yield surfaces and in the absence of imposed internal dislocations ($\dot{\boldsymbol{\delta}} = \mathbf{0}$) (see Section 8)

3.3.2. Assume the same restrictions (3.21) as in the preceding subsection. Cottle's symmetric duality theory of quadratic programming, as outlined in Appendix A.2, allows us to write immediately the dual of problem (3.23), simply by taking account in (A7) of the identifications (3.22):

$$\begin{aligned} &\text{maximize} && -\tfrac{1}{2}\tilde{\boldsymbol{\zeta}}\mathbf{M}\boldsymbol{\zeta} \\ &\text{subject to} && \mathbf{q}+\mathbf{M}\boldsymbol{\zeta} \geq \mathbf{0} \end{aligned} \tag{3.31}$$

Substitution into (3.31), written as a minimization problem, of equations (3.12), (3.13) and, subsequently, of equations (3.5), (3.8), (3.7) with $\mathbf{V} = \mathbf{N}$, leads, through trivial algebraic manipulations and addition of the immaterial constant $\frac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}\dot{\boldsymbol{\delta}}$, to the quadratic program:

$$\text{minimize} \quad Q_{\text{II}}(\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}) \equiv \tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + \tfrac{1}{2}(\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}}) - \underline{\tfrac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}\dot{\boldsymbol{\delta}}} \tag{3.32}$$

$$\text{subject to} \quad \tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\dot{\mathbf{u}} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N})\dot{\boldsymbol{\lambda}} \leq \tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{3.33}$$

$$(\mathbf{K}_E + \mathbf{K}_G)\dot{\mathbf{u}} - \tilde{\mathbf{B}}\mathbf{S}\mathbf{N}\dot{\boldsymbol{\lambda}} = \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}}. \tag{3.34}$$

This can be translated in the statement:

(III) *Under the hypothesis* (3.21), *among all distributions of displacement rates* $\dot{\mathbf{u}}$ *and plastic multiplier rates* $\dot{\boldsymbol{\lambda}}$ *which comply with the linear constraints* (3.33), (3.34), *the/any distribution which characterizes the actual response (if there is any) to given external action rates* $\dot{\mathbf{F}}$, $\dot{\boldsymbol{\delta}}$, *minimizes the quadratic function* (3.32).

Note that inequality (3.33) expresses the constitutive requirement that the plastic potentials nowhere become positive, equation (3.34) expresses equilibrium.

We derive below some consequent formulations which give a better mechanical insight in the above extremum properties and in their connexions with previous more particular results, though they do not offer computational advantages.

Substitute the expression (3.5) of $\mathbf{K}_E$ in (3.34) and take into account the equations:

$$\mathbf{B}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\delta}}, \qquad \dot{\boldsymbol{\varepsilon}}^e = \mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} \tag{3.35}$$

in (3.32)–(3.34). These become:

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\sigma}}}\mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} + \tilde{\dot{\boldsymbol{\sigma}}}\dot{\boldsymbol{\delta}} \tag{3.36}$$

$$\text{subject to} \quad \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}} - \mathbf{H}\dot{\boldsymbol{\lambda}} \leq \mathbf{0}, \qquad \mathbf{K}_G\dot{\mathbf{u}} + \tilde{\mathbf{B}}\dot{\boldsymbol{\sigma}} = \dot{\mathbf{F}} \tag{3.37a, b}$$

$$\mathbf{B}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}} = \mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\delta}} \tag{3.37c}$$

The optimization with respect to the constrained independent variables $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\sigma}}$ replaces the direct use of some constitutive rules ($\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$, $\tilde{\dot{\boldsymbol{\phi}}}\dot{\boldsymbol{\lambda}} = 0$) which are generally violated by the generic feasible choice. Note that compatibility is *not* a consequence of the optimization process, but must be explicitly assumed as a constraint set, equation (3.37c).

Therefore Theorem (III) can be neither formally nor conceptually regarded as a generalization of the complementary energy principle. However, *under the stronger assumption that both matrices* $\mathbf{K}_G$ *and* $\mathbf{H}$ *are positive semidefinite, the compatibility constraint* (3.37c) *becomes superfluous*. A proof of this fact can be outlined as follows: (3.36) becomes a convex function of the supervector $[\tilde{\mathbf{u}} \mid \tilde{\boldsymbol{\lambda}} \mid \tilde{\boldsymbol{\sigma}}]$; (3.36) (3.37a,b) may be reduced to the forms (A6) and, hence, (A3) with $\mathbf{E} = \mathbf{0}$, through simple equivalence transformations; the relation set of the kind (A3) thus obtained is readily seen to be equivalent to the system of the governing mechanical laws (2.8)–(2.11) with $\mathbf{V} = \mathbf{N}$ if $\mathbf{K}_G$ and $\mathbf{H}$ are definite, to admit all solutions of this system (and possibly others) if $\mathbf{K}_G$ and/or $\mathbf{H}$ are only semidefinite. In the former case dualization leads again to Theorem (II).

In the absence of large displacements ($\mathbf{K}_G = \mathbf{0}$) and softening ($\mathbf{H}$ positive semidefinite, Drucker's postulate valid), (3.36) becomes convex as a function of the supervector $[\tilde{\dot{\boldsymbol{\lambda}}} \mid \tilde{\dot{\boldsymbol{\sigma}}}]$. Under these hypotheses, we may formulate Theorem (III) simply as follows:

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\sigma}}}\mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} + \tilde{\dot{\boldsymbol{\sigma}}}\dot{\boldsymbol{\delta}} \tag{3.38}$$

$$\text{subject to} \quad \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}} - \mathbf{H}\dot{\boldsymbol{\lambda}} \leq \mathbf{0}; \qquad \tilde{\mathbf{B}}\dot{\boldsymbol{\sigma}} = \dot{\mathbf{F}}. \tag{3.39}$$

Let us prescribe that any trial $[\tilde{\dot{\boldsymbol{\lambda}}} \mid \tilde{\dot{\boldsymbol{\sigma}}}]$ respects the constitutive rate relations and generates through them $\dot{\boldsymbol{\varepsilon}}$; this does not affect the solution, and through (3.28) allows us to transform (3.38), (3.39) in:

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\boldsymbol{\sigma}}}(\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\delta}}) \quad \text{subject to} \quad \tilde{\mathbf{B}}\dot{\boldsymbol{\sigma}} = \dot{\mathbf{F}} \tag{3.40}$$

This is the expression, in the context of the matrix theory, of Mandel's generalization [12] of the *static* minimum principle of incremental plasticity (Prager–Hodge [14]). (3.38), (3.39) is the analogous generalization of a theorem proved in [17]. If $\mathbf{H}$ is not positive semidefinite, optimality conditions of the kind (A3) still coincide with the governing relations of the mechanical problem, but are only necessary, i.e. *not all* solutions of the mechanical problem minimize (3.38). This shows that the static extremum principle cannot remain valid in general when physical instabilizing effects are present, a conclusion already reached in [23].

4. EXTREMUM PRINCIPLES FOR PLASTIC MULTIPLIER RATES

4.1 *The* $\dot{\boldsymbol{\lambda}}, \boldsymbol{\varphi}$ *formulation*

We shall assume now that

$$\det|\mathbf{K}| \equiv \det|\tilde{\mathbf{B}}\mathbf{S}\mathbf{B} + \mathbf{K}_G| \neq 0. \tag{4.1}$$

On the basis of equations (3.5), (3.6) and relevant remarks (Section 3.1), hypothesis (4.1) physically means that there would not exist any other configuration adjacent to the configuration Y and equilibrated under the same external loading condition, if the system were to behave in a purely elastic manner in the neighbourhood of Y. Under assumption (4.1), equation (3.2) may be solved with respect to vector $\dot{\mathbf{u}}$:

$$\dot{\mathbf{u}} = \mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S}\mathbf{V}\dot{\boldsymbol{\lambda}} + \mathbf{K}^{-1}\dot{\mathbf{F}} - \mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{4.2}$$

Substitute $\dot{\mathbf{u}}$ in equation (3.3). Thus we obtain from (3.2)–(3.4), for the problem in hand, the following formulation involving only the unknown vectors $\dot{\boldsymbol{\lambda}}, \boldsymbol{\varphi}$:

$$\boldsymbol{\varphi} = -[\mathbf{H} - \tilde{\mathbf{N}}(\mathbf{S}\mathbf{B}\mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S} - \mathbf{S})\mathbf{V}]\dot{\boldsymbol{\lambda}} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\mathbf{K}^{-1}\dot{\mathbf{F}} + \tilde{\mathbf{N}}(\mathbf{S}\mathbf{B}\mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S} - \mathbf{S})\dot{\boldsymbol{\delta}} \tag{4.3}$$

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0}, \qquad \boldsymbol{\varphi} \leq \mathbf{0}, \qquad \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0 \tag{4.4a, b, c}$$

The following new symbols and relations are introduced and justified below:

$$\mathbf{Z}_G \equiv \mathbf{SBK}^{-1}\tilde{\mathbf{B}}\mathbf{S}-\mathbf{S} \tag{4.5}$$

$$\dot{\boldsymbol{\sigma}}^E \equiv \dot{\boldsymbol{\sigma}}^{EF}+\dot{\boldsymbol{\sigma}}^{E\delta} \equiv \mathbf{SBK}^{-1}\dot{\mathbf{F}}+\mathbf{Z}_G\dot{\boldsymbol{\delta}}. \tag{4.6}$$

The symmetric matrix $\mathbf{Z}_G$ defined by equation (4.5) is easily shown to be the operator which transforms any dislocation rate vector $\dot{\boldsymbol{\delta}}$ into the stress response $\dot{\boldsymbol{\sigma}}^{E\delta}$ to $\dot{\boldsymbol{\delta}}$ in the supposedly purely elastic regime of large displacement. In fact, if plastic yielding is ruled out and (4.1) holds, the displacement rates due to $\dot{\boldsymbol{\delta}}$ are supplied by equation (3.6) for $\dot{\mathbf{F}} = \mathbf{0}$:

$$\dot{\mathbf{u}}^\delta = \mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{4.7}$$

whence, through the compatibility equation (2.10) and the elastic stress–strain relation, we obtain:

$$\dot{\boldsymbol{\sigma}}^{E\delta} = \mathbf{S}(\mathbf{B}\dot{\mathbf{u}}^\delta-\dot{\boldsymbol{\delta}}) = (\mathbf{SBK}^{-1}\tilde{\mathbf{B}}\mathbf{S}-\mathbf{S})\dot{\boldsymbol{\delta}}. \tag{4.8}$$

This justifies both the above remark on the meaning of $\mathbf{Z}_G$ and the symbols introduced in (4.6) and clarifies their meanings: $\dot{\boldsymbol{\sigma}}^E$ represents the elastic stress rate response to the whole set of straining effect rates and is supplied by a preliminary calculation, centered on equation (3.6) and, hence, linear even in the presence of large displacements.

4.2 *A "general" minimum theorem for plastic multiplier rates*

Using equations (4.5) and (4.6) we may express the $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$ formulation (4.3), (4.4) in a more compact form:

$$-\dot{\boldsymbol{\varphi}} = (\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{V})\dot{\boldsymbol{\lambda}}-\tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}^E \tag{4.9a}$$

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0}, \qquad -\dot{\boldsymbol{\varphi}} \geq \mathbf{0}, \qquad \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0. \tag{4.9b, c, d}$$

If compared to (A.1) (see Appendix), the relation set (4.9) in $\dot{\boldsymbol{\varphi}}$, $\dot{\boldsymbol{\lambda}}$ immediately reveals the mathematical structure of a linear complementarity problem defined by the data:

$$\mathbf{M}^* \equiv \mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{V}; \qquad \mathbf{q}^* = \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}^E. \tag{4.10a, b}$$

The equivalent quadratic program, patterned according to (A.2), gives rise to the following theorem:

(IV) *In the class of all sets of plastic multiplier rates which fulfil the linear constraints:*

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0} \tag{4.11}$$

$$(\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{V})\dot{\boldsymbol{\lambda}}-\tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}^E \geq \mathbf{0} \tag{4.12}$$

the/any solution set, if any, minimizes the quadratic functional

$$R(\dot{\boldsymbol{\lambda}}) \equiv \tilde{\dot{\boldsymbol{\lambda}}}(\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{V})\dot{\boldsymbol{\lambda}}-\tilde{\dot{\boldsymbol{\sigma}}}^E\mathbf{N}\dot{\boldsymbol{\lambda}}. \tag{4.13}$$

Conversely, the minimum of $R(\dot{\boldsymbol{\lambda}})$, if zero, characterizes the solution; if not zero the incremental problem admits no solution.

The generality of this statement is limited only by the hypothesis (4.1) $\det|\mathbf{K}| \neq 0$; symmetry and definiteness of matrix $\mathbf{M}^*$ (and hence convexity of R) are not required, so that allowance is made for any degree and kind of instability covered by "linear" plastic flow-laws or due to second-order geometrical effects. When small displacements and non-interacting yielding modes are hypothesized, Theorem (IV) specializes to the minimum principle established in [18], in tensor notation, for continua with nonassociated flow-laws. Other minimum properties for $\dot{\boldsymbol{\lambda}}$ cannot be expected at the same level of generality due to the self-dual character of quadratic programs of kind (A.2) (see Appendix).

4.3 "*Particular*" *extremum theorems for plastic multiplier rates*

Assume now that matrix $\mathbf{M}^*$, equation (4.10a), is *symmetric positive semidefinite* [hypothesis (4.14)]. This implies *normality* ($\mathbf{V} = \mathbf{N}$) and *reciprocal* interaction ($\tilde{\mathbf{H}} = \mathbf{H}$), and represents a restrictive condition on the cumulative effects of possible softening and geometry changes. Starting from the above hypothesis (4.14), we shall follow the same path of reasoning as in Section 3: namely, the complementarity problem (4.9) is interpreted as sufficient and necessary local Kuhn–Tucker conditions by comparing it to the relation set (A.3) and identifying:

$$\mathbf{A} = \mathbf{0}, \qquad \mathbf{E} = \mathbf{0}, \qquad \mathbf{D} = \mathbf{M}^*, \qquad \mathbf{c} = \mathbf{q}^*, \qquad \mathbf{b} = \mathbf{0}. \tag{4.15}$$

Thereafter the equivalent quadratic program (A.6) and its dual (A.7) are readily formed using again (4.15), and can be expressed in the statements which follow:

(V) *When the hypotheses* (4.1) *and* (4.14) ($\mathbf{M}^*$ *nonnegative definite*) *hold, within the class of all nonnegative plastic multiplier rate sets,*

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0} \tag{4.16}$$

the/any solution set, if any, maximizes the quadratic function:

$$R_{\mathrm{I}}(\dot{\boldsymbol{\lambda}}) \equiv -\tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}(\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{N})\dot{\boldsymbol{\lambda}} + \tilde{\dot{\boldsymbol{\sigma}}}^E\tilde{\mathbf{N}}\dot{\boldsymbol{\lambda}}. \tag{4.17}$$

(VI) *Under the hypothesis* (4.1) *and* (4.14), *in the class of all distributions of plastic multiplier rates which satisfy the linear inequality:*

$$(\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{N})\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}^E \geq \mathbf{0} \tag{4.18}$$

the/any solution, if any, minimizes the quadratic function:

$$R_{\mathrm{II}}(\dot{\boldsymbol{\lambda}}) \equiv \tfrac{1}{2}\tilde{\dot{\boldsymbol{\lambda}}}(\mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{N})\dot{\boldsymbol{\lambda}}. \tag{4.19}$$

If the additional restrictions of small displacements and noninteracting yielding modes are imposed, Theorem (V) reduces to Ceradini's principle [19], and Theorem (VI) to a minimum principle established in [20].

In the range of validity of Theorems (V) and (VI), Theorem (IV) is a straightforward consequence of them, since (4.11)–(4.13) represents the "composite program", in Cottle's sense (see Appendix A.3), of programs (4.16), (4.17) and (4.18), (4.19).

All the extremum properties (IV)–(VI) presume that both equilibrium and compatibility be *a priori* fulfilled: the constraints (4.11), (4.16) and (4.12), (4.18) express constitutive inequalities (the latter requires that the yield functions do not become positive); the optimization process is equivalent to explicitly imposing the remaining flow rules, specifically the nonlinear complementarity condition.

Finally it is worth noting the correspondence between Theorems (I)–(III) on one side and Theorems (IV)–(VI) respectively, on the other: in fact an alternative derivation of the latter theorems might be obtained simply by substituting in the analytical formulations of the former theorems the expression (4.2) for the displacements rates.

5. BOUNDS TO THE INSTANTANEOUS COMPLIANCE WITH RESPECT TO A SINGLE LOAD OR DISLOCATION

5.1 *Preliminaries*

It may be useful, especially in the presence of large displacements, to evaluate or estimate in a given static situation Y the ratio between a single load increment δF_h applied to a nodal point of the system and the incremental displacement δu_h^0 it causes there in its direction. We shall call this ratio and its inverse (local instantaneous) stiffness and compliance (or flexibility) χ_h^F respectively:

$$\chi_h^F = \frac{\dot{u}_h^0}{\dot{F}_h}. \tag{5.1}$$

Analogous notions can be introduced for a single dislocation component $\dot{\delta}_h$:

$$\chi_h^\delta = -\frac{\dot{\delta}_h}{\dot{\sigma}_h^0} \tag{5.2}$$

where $\dot{\sigma}_h^0$ indicates the corresponding stress component generated by $\dot{\delta}_h$ and the minus is required by the identity of sign conventions for δ_h and σ_h.

In order to develop operative criteria for bounding χ_h^F and χ_h^δ the following corollaries of the preceding theorems are first proved.

(VIIa) *For load increments only* ($\dot{\boldsymbol{\delta}} = \mathbf{0}$), *the (extremum) values assumed at solution by the functions Q_{I} and Q_{II} to be optimized according to Theorems (II) and (III), equal (to within δt^2) the work performed by the given load increments for the displacements they cause.*

(VIIb) *For dislocation increments only* ($\dot{\mathbf{F}} = \mathbf{0}$), *the above optimal values equal, to within $-\delta t^2$, the work performed by the consequent incremental stresses for the given dislocation increments.*

Proof. The energy function considered by the general Theorem (I), Section 3.2, is related to those of Theorems (II) and (III) through:

$$Q = Q_{\mathrm{II}} - Q_{\mathrm{I}} \tag{5.3}$$

and vanishes at any solution $\dot{\mathbf{u}}^0, \dot{\boldsymbol{\lambda}}^0$. Hence:

$$Q_{\mathrm{I}}(\dot{\mathbf{u}}^0, \dot{\boldsymbol{\lambda}}^0) = Q_{\mathrm{II}}(\dot{\mathbf{u}}^0, \dot{\boldsymbol{\lambda}}^0) \tag{5.4}$$

as the duality theory shows in general [see Appendix A.3(a)]. Taking into account in equation (5.4) the expressions (3.25), (3.32), and the equation

$$\mathbf{B}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}} - \dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\varepsilon}}^e = \mathbf{S}^{-1}\dot{\boldsymbol{\sigma}} \tag{5.5}$$

which derives from (2.10), (2.8), (2.3), we obtain:

$$Q_{\mathrm{I}}(\dot{\mathbf{u}}^0, \dot{\boldsymbol{\lambda}}^0) = Q_{\mathrm{II}}(\dot{\mathbf{u}}^0, \dot{\boldsymbol{\lambda}}^0) = \tfrac{1}{2}\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}}^0 + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\dot{\boldsymbol{\sigma}}^0. \tag{5.6}$$

Statement (a) flows from equation (5.6) for $\dot{\boldsymbol{\delta}} = \mathbf{0}$, statement (b) for $\dot{\mathbf{F}} = \mathbf{0}$.

5.2 *Upper and lower bounds from feasible* $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}$ *distributions*

In a constrained optimization a variables set is called "feasible" if it satisfies the constraints. We shall indicate by $\dot{\mathbf{u}}', \dot{\boldsymbol{\lambda}}'$ and $\dot{\mathbf{u}}'', \dot{\boldsymbol{\lambda}}''$ the vectors $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}$ which fulfil the constraints (3.24) of Theorem (II) and (3.33), (3.34) of Theorem (III), respectively.

Vectors denoted by $\dot{\mathbf{u}}'''$, $\dot{\boldsymbol{\lambda}}'''$ shall form the intersection of these two feasible classes, i.e. the class of vectors which are feasible with respect to all the above constraints and, hence, to the programming problem postulated by Theorem (I).

Suppose that, at a given stage Y of the loading history of the system, hypothesis (3.21), Section 3.3, holds, so that the validity of Theorems (II) and (III) is ensured.

(VIII) *The compliance* χ_h^F *exhibited by the system with respect to a nodal load component h, can be bracketed as follows, if vectors of the above feasible classes are obtained on the basis of a load rate* $\dot{F}_h$ *of arbitrary magnitude*:

$$\frac{2}{\dot{F}_h^2}Q_{\mathrm{I}}(\dot{\mathbf{u}}', \dot{\boldsymbol{\lambda}}') \leq \chi_h^F \leq \frac{2}{\dot{F}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}'', \dot{\boldsymbol{\lambda}}'') \tag{5.7}$$

$$\frac{2}{\dot{F}_h^2}Q_{\mathrm{I}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}''') = \frac{2\dot{u}_h'''}{\dot{F}_h} - \frac{2}{\dot{F}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}''') \leq \chi_h^F \leq \frac{2}{\dot{F}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}''') \tag{5.8}$$

Proof. Rewrite (5.1) in the form

$$\chi_h^F = \frac{\frac{1}{2}\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}}^0}{\frac{1}{2}\tilde{\dot{\mathbf{F}}}\dot{\mathbf{F}}} \tag{5.9}$$

where $\tilde{\dot{\mathbf{F}}} \equiv [0 \ldots \dot{F}_h \ldots 0]$ and $\dot{\mathbf{u}}^0$ is the exact displacement rate response to $\dot{\mathbf{F}}$.

If we compare equation (5.9) to (5.6) for $\dot{\boldsymbol{\delta}} = \mathbf{0}$ and apply Theorems (II) and (III), the continued inequalities (5.7) and (5.8) immediately follow through the definitions of the feasible vectors involved. If a vector pair $\dot{\mathbf{u}}'''$, $\dot{\boldsymbol{\lambda}}'''$ is available, it supplies both a lower and an upper bound; for the former, the second expression in (5.8) supplies a useful alternative form, which derives from the fact that quadratic terms in the expressions of Q_{I} (3.25) and Q_{II} (3.32) are equal for equal arguments.

(IX) *The stiffness* $(\chi_h^\delta)^{-1}$ *which the system in the given configuration exhibits with respect to the* hth *dislocation component, can be bracketed by means of the following inequalities, as soon as feasible vectors are available with respect to an arbitrary* $\dot{\delta}_h$:

$$-\frac{2}{\dot{\delta}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}'', \dot{\boldsymbol{\lambda}}'') \leq (\chi_h^\delta)^{-1} \leq -\frac{2}{\dot{\delta}_h^2}Q_{\mathrm{I}}(\dot{\mathbf{u}}', \dot{\boldsymbol{\lambda}}') \tag{5.10}$$

$$-\frac{2}{\dot{\delta}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}) \leq (\chi_h^\delta)^{-1} \leq -\frac{2}{\dot{\delta}_h^2}Q_{\mathrm{I}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}''') = -\frac{2}{\dot{\delta}_h^2}\tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}(\mathbf{B}\dot{\mathbf{u}}''' - \mathbf{N}\dot{\boldsymbol{\lambda}}''' - \dot{\boldsymbol{\delta}}) + \frac{2}{\dot{\delta}_h^2}Q_{\mathrm{II}}(\dot{\mathbf{u}}''', \dot{\boldsymbol{\lambda}}''') \tag{5.11}$$

where $\tilde{\dot{\boldsymbol{\delta}}} \equiv [0 \ldots \dot{\delta}_h \ldots 0]$.

The proof of these inequalities can be patterned on the preceding one, by rewriting (5.2) as

$$(\chi_h^\delta)^{-1} = -\frac{\frac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\dot{\boldsymbol{\sigma}}^0}{\frac{1}{2}\tilde{\dot{\boldsymbol{\delta}}}\dot{\boldsymbol{\delta}}} \tag{5.12}$$

and making use again of (5.6) for $\dot{\mathbf{F}} = \mathbf{0}$.

All the above inequalities become equalities if the feasible vectors represent exact solutions for the external action component $\dot{F}_h$ or $\dot{\delta}_h$ considered. Since feasible vectors are much more easily found than optimal vectors, (5.7), (5.8) and (5.10), (5.11) are useful for estimates of local flexibilities.

It is possible to derive parallel inequalities which require the knowledge of vectors feasible with respect to the quadratic programs of Theorems (IV)–(VI) and to bracket the difference between the actual and the purely elastic local compliance with respect to $\dot{F}_h$ or stiffness with respect to $\dot{\delta}_h$. These inequalities will not be derived here, for the sake of brevity. Bounds of this kind were established in [24] in the narrower context of small displacement theory of frames.

In conclusion, it is important to keep in mind that the finite element models referred to herein, lead in general to underestimating the flexibility of the continuous system with respect to a single load (the discretization adopted is in line with the displacement methods, and satisfies compatibility everywhere, and equilibrium at the nodes [25]). Clearly the bounds established above, concern the true flexibility *of the model*, not that of the original continuum, which in turn is bounded from below by that of the model.

6. ON UNIQUENESS OF SOLUTIONS

6.1 *General problems*

By "general" we mean without any restrictions on the nature or amount of the instabilizing effects. The following statements can be readily derived from the preceding analysis combined with suitable mathematical notions.

(X) *Provided that* $\det|\mathbf{K}| \neq 0$, (6.1) [*or* (4.1)], *the solution exists and is unique for any set of external action rates, if and only if the matrix*

$$\mathbf{M}^* \equiv \mathbf{H}-\tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{V} \quad \text{where} \quad \mathbf{Z}_G \equiv \mathbf{S}\mathbf{B}\mathbf{K}^{-1}\tilde{\mathbf{B}}\mathbf{S}-\mathbf{S} \tag{6.2}$$

has all principal subdeterminants positive (*i.e. it is a P-matrix*).

As already noted in Section 4.1, the circumstance (6.1) means response uniqueness (and existence) for hypothetical purely elastic behaviour in the neighbourhood of the current situation considered; this elastic behaviour can be analytically prescribed by assuming $\mathbf{H} = k\mathbf{I}$, with $k \to \infty$, where $\mathbf{I}$ is the identity matrix and k a scalar. If $\det|\mathbf{K}| = 0$, $\mathbf{Z}_G$ and hence $\mathbf{M}^*$ would become meaningless.

When the plastic multipliers $\dot{\boldsymbol{\lambda}}$ are known, they uniquely determine, through (4.2), the velocities, whence, through (3.1), the stress rates, and through (2.8) the strain rates. Therefore uniqueness of $\dot{\boldsymbol{\lambda}}$ implies uniqueness of the whole incremental response of the system. In Section 4.1, the search for $\dot{\boldsymbol{\lambda}}$ has been reduced to the linear complementarity problem (4.3), (4.4): therefore statement (X) follows immediately from theorem (i) of Appendix A.5 applied to the $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$ formulation (4.3), (4.4).†

For the same reasons we obtain from Theorem (ii) of Appendix A.5, the statement:

(XI) *When* $\det|\mathbf{K}| \neq 0$, *the number of solutions is finite for all external action rate sets, if, and only if, all the principal subdeterminants of matrix* (6.2) *are nonzero.*

(XII) *If* $\det|\mathbf{K}| \neq 0$, *the positive definiteness of matrix* $\mathbf{M}^*$ (6.2) *is a sufficient condition for the solution to exist and be unique for any set of external action rates.*

This is an immediate consequence of (X) since the class of all P-matrices contains as a subclass the positive definite matrices.

† The necessity of this and of the subsequent statements (XI) and (XV) is clarified by the following remark. It appears from (4.3) via (4.5), (4.6) that the known vector of the complementarity problem in question is $\mathbf{q} = \tilde{\mathbf{N}}\dot{\boldsymbol{\sigma}}^E$. As the vectors $\mathbf{N}_j^i$ at each corner are linearly independent, matrix $\mathbf{N}$ has full column rank: therefore it is possible to find a vector $\dot{\boldsymbol{\sigma}}^E$ corresponding to any $\mathbf{q}$. From (2.3), (2.10), (2.11) we obtain $\mathbf{S}^{-1}\dot{\boldsymbol{\sigma}}^E = \mathbf{B}\dot{\mathbf{u}}-\dot{\boldsymbol{\delta}}$, $-\tilde{\mathbf{B}}\dot{\boldsymbol{\sigma}}^E = \mathbf{K}_G\dot{\mathbf{u}}-\dot{\mathbf{F}}$, which show that for any stress rate distribution $\dot{\boldsymbol{\sigma}}^E$ there exist external action rates $\dot{\mathbf{F}}$, $\dot{\boldsymbol{\delta}}$ capable of generating $\dot{\boldsymbol{\sigma}}^E$ in elastic conditions.

(XIII) *When* det$|\mathbf{K}| \neq 0$, *and* $\mathbf{M}^*$ (6.2) *is positive semidefinite, if there exists a nonnegative plastic multiplier vector* $\dot{\boldsymbol{\lambda}}$ *which makes the yield function nonpositive* [*i.e. a* $\dot{\boldsymbol{\lambda}}$ *feasible with respect to the constraints* (4.11) *and* (4.12)], *then a solution of the incremental problem exists.* This statement is simply the application of Cottle's Theorem (iii) (Appendix A.5) to the complementarity problem (4.9).

Starting from the $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{\varphi}}$ formulation, of Section 3, the restriction det$|\mathbf{K}| \neq 0$ is not needed, but the matrix $\mathbf{M}$ of the equivalent complementarity problem, by its very nature, Equation (3.13), can be neither a P-matrix, nor definite. However it is still possible to obtain the following sufficient criterion of uniqueness:

(XIV) *The solution is unique, for any set of external action rates, if the matrix*

$$\begin{bmatrix} \mathbf{K} & -\mathbf{C}_V \\ -\tilde{\mathbf{C}}_N & \mathbf{G} \end{bmatrix} \text{ [cf. (3.7), (3.8)] } \textit{is positive definite} \tag{6.3}$$

or, in other terms, if

$$\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + (\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\bar{\mathbf{B}}\dot{\mathbf{u}} - \mathbf{V}\dot{\boldsymbol{\lambda}}) > \mathbf{0} \tag{6.4}$$

for any vector $[\tilde{\dot{\mathbf{u}}} \,\vdots\, \tilde{\dot{\boldsymbol{\lambda}}}] \neq \mathbf{0}$.

Proof. Suppose that the complementarity problem (3.14), (3.15) has two solutions $\boldsymbol{\zeta}_1, \boldsymbol{\omega}_1$ and $\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1 + \Delta\boldsymbol{\zeta}$, $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \Delta\boldsymbol{\omega}$, for the same vector $\mathbf{q}$. Then

$$\Delta\boldsymbol{\omega} = \mathbf{M}\Delta\boldsymbol{\zeta} \tag{6.5}$$

whence premultiplying by $\Delta\tilde{\boldsymbol{\zeta}}$:

$$\tilde{\boldsymbol{\omega}}_1\boldsymbol{\zeta}_1 + \tilde{\boldsymbol{\omega}}_2\boldsymbol{\zeta}_2 - \tilde{\boldsymbol{\omega}}_1\boldsymbol{\zeta}_2 - \tilde{\boldsymbol{\omega}}_2\boldsymbol{\zeta}_1 = \Delta\tilde{\boldsymbol{\zeta}}\mathbf{M}\Delta\boldsymbol{\zeta} \tag{6.6}$$

The r.h.s. of (6.6), through (3.12), (3.13) becomes the quadratic form in $[\Delta\tilde{\dot{\mathbf{u}}} \,\vdots\, \Delta\tilde{\dot{\boldsymbol{\lambda}}}]$ associated with matrix (6.3), and precisely:

$$\Delta\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\Delta\dot{\mathbf{u}} + \Delta\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\Delta\dot{\boldsymbol{\lambda}} + (\Delta\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \Delta\tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\Delta\dot{\mathbf{u}} - \mathbf{V}\Delta\dot{\boldsymbol{\lambda}}) \tag{6.7}$$

The expression (6.7) must be positive, if (6.4) holds, unless the differences $\Delta\dot{\mathbf{u}}, \Delta\dot{\boldsymbol{\lambda}}$ vanish. Since the l.h.s. of (6.6) is nonpositive because of (3.14b) and (3.15a), equation (6.6) requires that (6.7), and hence $\Delta\dot{\mathbf{u}}, \Delta\dot{\boldsymbol{\lambda}}$, be zero (q.e.d.).

6.2 *Problems with symmetric operators*

In less general cases, with normality ($\mathbf{V} = \mathbf{N}$) and reciprocity interaction ($\tilde{\mathbf{H}} = \mathbf{H}$), all matrix operators involved become symmetric ($\tilde{\mathbf{M}}^* = \mathbf{M}^*$, $\tilde{\mathbf{M}} = \mathbf{M}$). This gives rise to further statements or to useful specializations of the preceding ones. Within the class of all symmetric matrices, the subsets of the positive definite and the P-matrices are coincident.

Hence Theorems (X) and (XII) can be replaced by a single statement:

(XV) *When* det$|\mathbf{K}| \neq 0$, *a unique solution exists for any set of external actions if, and only if, the matrix*

$$\mathbf{M}^* \equiv \mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}_G\mathbf{N} \tag{6.8}$$

is positive definite.

Situations characterized by *positive semidefiniteness* of matrix (6.3) and, hence, of matrix $\mathbf{M}$, are particularly interesting when $\tilde{\mathbf{V}} = \mathbf{N}$ (so that $\mathbf{C}_V = \mathbf{C}_N = \mathbf{C}$) and $\tilde{\mathbf{H}} = \mathbf{H}$. In these cases the minimum principle (II) holds and the theorem of Appendix A.4 can be used in order to define the totality of relevant solutions. With reference to (3.23) this theorem requires that the difference $\Delta\zeta$ between two solutions satisfies the equations:

$$\Delta\tilde{\zeta}\mathbf{M}\Delta\zeta = 0 \tag{6.9}$$

$$\tilde{\mathbf{q}}\Delta\zeta = 0. \tag{6.10}$$

Substituting (3.7), (3.8), (3.10), (3.12), (3.13) we obtain from (6.9) and (6.10) respectively:

$$\Delta\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\Delta\dot{\mathbf{u}} + \Delta\tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\Delta\dot{\boldsymbol{\lambda}} + (\Delta\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \Delta\tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\Delta\dot{\mathbf{u}} - \mathbf{N}\Delta\dot{\boldsymbol{\lambda}}) = 0 \tag{6.11}$$

$$\tilde{\dot{\mathbf{F}}}\Delta\dot{\mathbf{u}} + \tilde{\dot{\boldsymbol{\delta}}}\mathbf{S}(\mathbf{B}\Delta\dot{\mathbf{u}} - \mathbf{N}\Delta\dot{\boldsymbol{\lambda}}) = 0. \tag{6.12}$$

The totality of solutions to the incremental problem can be expressed as

$$\dot{\mathbf{u}}^0 = \bar{\dot{\mathbf{u}}}^0 + \Delta\dot{\mathbf{u}}^0; \qquad \dot{\boldsymbol{\lambda}}^0 = \bar{\dot{\boldsymbol{\lambda}}}^0 + \Delta\dot{\boldsymbol{\lambda}}^0 \tag{6.13}$$

$$\text{subject to} \qquad \dot{\boldsymbol{\lambda}}^0 \geq \mathbf{0} \tag{6.14}$$

where $\bar{\dot{\mathbf{u}}}^0, \bar{\dot{\boldsymbol{\lambda}}}^0$ represent a fixed solution (if any), and $\Delta\dot{\mathbf{u}}^0, \Delta\dot{\boldsymbol{\lambda}}^0$ represent any solution to the equation set (6.11), (6.12).

Assume $\dot{\mathbf{F}} = \mathbf{0}, \dot{\boldsymbol{\delta}} = \mathbf{0}$: a solution is certainly $\bar{\dot{\mathbf{u}}}^0 = \mathbf{0}, \bar{\dot{\boldsymbol{\lambda}}}^0 = \mathbf{0}$, and (6.12) becomes trivially satisfied. Two cases can be distinguished with reference to (6.11):

(i) equation (6.11) admits a solution $\Delta\dot{\mathbf{u}}^{0\prime}, \Delta\dot{\boldsymbol{\lambda}}^{0\prime} \geq \mathbf{0}$. In this case, *in the neighbourhood of the configuration Y there is an unbounded set of adjacent configurations equilibrated under the same external actions* (neutral equilibrium of Y or "eigenstate", in Hill's terminology [2]). In fact, for $\dot{\mathbf{F}} = \mathbf{0}, \dot{\boldsymbol{\delta}} = \mathbf{0}$, (6.12) is satisfied; $\bar{\dot{\mathbf{u}}}^0 = \mathbf{0}, \bar{\dot{\boldsymbol{\lambda}}}^0 = \mathbf{0}$ is a solution and the solution set $\dot{\mathbf{u}}^0 = \alpha\Delta\dot{\mathbf{u}}^{0\prime}, \dot{\boldsymbol{\lambda}}^0 = \alpha\Delta\dot{\boldsymbol{\lambda}}^{0\prime}$ complies with (6.14) for any $\alpha > 0$. In other terms (6.11) defines a *feasible ray* in the $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}}$ space where the optimization, according to Theorem (II) is to be performed.

(ii) Let all $\Delta\dot{\boldsymbol{\lambda}}^0$ solutions of (6.11) contain both positive and negative components. Then *the only possible solution for* $\dot{\mathbf{F}} = \mathbf{0}, \dot{\boldsymbol{\lambda}} = \mathbf{0}$ *is the trivial one* ($\dot{\mathbf{u}}^0 = \mathbf{0}, \dot{\boldsymbol{\lambda}}^0 = \mathbf{0}$). Assume that for given external action rates $\dot{\mathbf{F}}, \dot{\boldsymbol{\delta}}$ there exists the optimal vector $\bar{\dot{\mathbf{u}}}^0, \bar{\dot{\boldsymbol{\lambda}}}^0$ and that $\Delta\dot{\mathbf{u}}^{0\prime}, \Delta\dot{\boldsymbol{\lambda}}^{0\prime}$ is a particular solution of (6.11), (6.12): then the constraint (6.14) defines along the straight line

$$\dot{\mathbf{u}}^0 = \bar{\dot{\mathbf{u}}}^0 + \alpha\Delta\dot{\mathbf{u}}^{0\prime}, \qquad \dot{\boldsymbol{\lambda}}^0 = \bar{\dot{\boldsymbol{\lambda}}}^0 + \alpha\Delta\dot{\boldsymbol{\lambda}}^{0\prime} \tag{6.15}$$

an interval $\overline{\alpha_{\min}\alpha_{\max}}$ which is necessarily *bounded*. In the present case an *infinite number of all bounded solutions are possible, i.e. an infinite number of incremental processes starting from Y may correspond, as alternatives, to the same* $\dot{\mathbf{F}}, \dot{\boldsymbol{\delta}}$. In a customary (Poincaré's) terminology this corresponds to a *bifurcation* (of the equilibrium path in the configuration space) without neutral equilibrium at Y: an occurrence first recognized by Shanley [1], and rigorously discussed later by Hill and Sewell [26] and Hill [2].

It is worth noting that the same conclusions can be reached on the basis of Theorem (V) for plastic multiplier rates.

As matrix $\mathbf{M}$ is symmetric positive semidefinite, equation (6.9) is fully equivalent to

$$\mathbf{M}\Delta\boldsymbol{\zeta} = \mathbf{0}. \tag{6.16}$$

Through (3.7), (3.8), (3.10), (3.12), (3.13), equation (6.16) yields two relations:

$$\mathbf{K}\Delta\dot{\mathbf{u}} - \tilde{\mathbf{B}}\mathbf{S}\mathbf{N}\Delta\dot{\boldsymbol{\lambda}} = \mathbf{0} \tag{6.17}$$

$$\tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\Delta\dot{\mathbf{u}} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N})\Delta\dot{\boldsymbol{\lambda}} = \mathbf{0} \tag{6.18}$$

whence, via (3.5), (2.8), (2.9a):

$$\mathbf{K}_G\Delta\dot{\mathbf{u}} + \tilde{\mathbf{B}}\Delta\dot{\boldsymbol{\sigma}} = \mathbf{0} \tag{6.19}$$

$$\tilde{\mathbf{N}}\Delta\dot{\boldsymbol{\sigma}} - \mathbf{H}\Delta\dot{\boldsymbol{\lambda}} = \Delta\dot{\boldsymbol{\phi}} = \mathbf{0}. \tag{6.20}$$

Equation (6.19) shows that the difference between two stress rate solutions is selfequilibrated for the velocity difference; the less obvious circumstance that *the yield function rates in the plastic elements are equal in all solutions*, appears from equation (6.20).

The l.h.s. of (6.11) coincides with the quadratic form of (6.4) when $\mathbf{V} = \mathbf{N}$ and $\tilde{\mathbf{H}} = \mathbf{H}$ are assumed in it. Thus the above discussion yields also an additional proof of the sufficiency of the uniqueness criterion (6.4) for the present narrower range. In this range the condition (6.4) can be reformulated as follows:

$$\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + \tilde{\dot{\boldsymbol{\varepsilon}}}^e\mathbf{S}\dot{\boldsymbol{\varepsilon}}^e > 0 \tag{6.21}$$

$$\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tilde{\dot{\boldsymbol{\sigma}}}\dot{\boldsymbol{\varepsilon}} > 0 \tag{6.22}$$

by means of the same considerations which led from (3.25) to (3.26) and (3.29). The latter form (6.22) is the version in the present approach of Hill's uniqueness criterion [2] in its first form.

6.3 *The extremum properties* (III) *and* (VI) *in the absence of solution uniqueness*

The extremum principles (III) and (VI) were derived in Section 3 and 4 by dualizing the quadratic programming problems which express the principle (II) and (V).

The duality Theorem (b) of Appendix A.3 ensures that *any* solution of the primal problem and, hence, of the incremental problem in hand, has the maximum properties (III) and (VI); but it does not guarantee that these maxima are attained only for solutions to the mechanical problem. Therefore, in view of the cases of nonuniqueness, it is useful to compare the totality (D) of solutions of (3.31) to the totality (P) of solutions of (3.23).

Let $\bar{\boldsymbol{\zeta}}^0$ be an optimal vector for (3.23) and, hence, also for (3.31); by virtue of the theorem quoted in Appendix A.4, the optimal vector sets (P) and (D) can be defined as:

$$(P)\ \boldsymbol{\zeta}^P - \bar{\boldsymbol{\zeta}}^0 + \Delta\boldsymbol{\zeta}^P \quad \text{with} \quad \mathbf{M}\Delta\boldsymbol{\zeta}^P = \mathbf{0} \tag{6.23}$$

$$\tilde{\mathbf{q}}\Delta\boldsymbol{\zeta}^P = 0 \tag{6.24}$$

$$\boldsymbol{\zeta}^P \geq \mathbf{0} \tag{6.25}$$

$$(D)\ \boldsymbol{\zeta}^D - \bar{\boldsymbol{\zeta}}^0 + \Delta\boldsymbol{\zeta}^D \quad \text{with} \quad \mathbf{M}\Delta\boldsymbol{\zeta}^D = \mathbf{0} \tag{6.26}$$

$$\mathbf{q} + \mathbf{M}\boldsymbol{\zeta}^D \geq \mathbf{0}. \tag{6.27}$$

We note that (6.27) is certainly verified because $\Delta\boldsymbol{\zeta}^D$ satisfies (6.26) and $\bar{\boldsymbol{\zeta}}^0$ solves (3.31). Moreover, (6.24) is certainly fulfilled by any $\Delta\boldsymbol{\zeta}^D$: in fact for two generic members of (D), say ζ_1^D, ζ_2^D, we may write

$$-\tilde{\mathbf{q}}\zeta_1^D = \tilde{\zeta}_1^D\mathbf{M}\zeta_1^D, \quad -\tilde{\mathbf{q}}\zeta_2^D = \tilde{\zeta}_2^D\mathbf{M}\zeta_2^D \tag{6.28}$$

since for all optimal vectors the objective functions of (3.23) and (3.31) are equal; it follows from (6.28) that:

$$-\tilde{\mathbf{q}}\Delta\zeta^D = \tilde{\zeta}_1^D\mathbf{M}\zeta_1^D - \tilde{\zeta}_2^D\mathbf{M}\zeta_2^D = \Delta\tilde{\zeta}^D\mathbf{M}\Delta\zeta^D + 2\tilde{\zeta}_2^D\mathbf{M}\Delta\zeta^D$$

which vanishes by virtue of (6.26).

Thus the conclusion is reached *that the whole set of* (P) *solutions to the incremental problem, can be obtained from the totality of solutions supplied by the dual principles* (III) *or* (VI), *simply by imposing the sign constraint* (6.25), *i.e.* $\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$.

7. STABILITY ANALYSIS

In plastic solids the changes in stresses and, hence, in internal energy are strongly path-dependent. A completely satisfactory answer regarding the stability of a given equilibrium configuration can be supplied, therefore, by a kinetic criterion: stability means, accordingly, that, in the motion produced by any set of perturbing forces acting in a time interval Δt, amplitudes and velocities remain bounded (asymptotically in time) and tend to vanish as the perturbing forces tend to vanish.

The clear difficulty of applying this criterion suggests recourse to so-called criteria of stability "in statical sense", which consider only straight line infinitesimal deformation paths going out from the equilibrium situation. However "statical" stability of path-dependent system is a rather controversial and ill-defined notion: critical considerations can be found in [2, 4, 13, 27–29]. We shall adopt here Drucker's statical stability criterion [13, 30] according to which a system in a state Y is *stable* if

$$L = \tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}} > 0 \tag{7.1}$$

for *any* transition $\dot{\mathbf{u}}\delta t$ from Y to a neighbouring configuration, $\dot{\mathbf{F}}\delta t$ being the external force increments required to ensure equilibrium in the disturbed state; the equilibrium at Y is called *neutral* if $L = 0$ for some paths and > 0 for all the others, *unstable* if there are some paths for which $L < 0$.

When the external forces are conservative, (7.1) seems to be sufficient for the kinematic stability of plastic systems, and perhaps also necessary when the flow-laws are associated (on the contrary, kinematically stable systems in the absence of normality may violate (7.1), as pointed out by Mandel [28]).

On the basis of the above Drucker's stability definition, the preceding analysis gives rise easily to the following remarks:

(XVI) *The system is stable in* Y *if*

$$\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tilde{\dot{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} + (\tilde{\dot{\mathbf{u}}}\tilde{\mathbf{B}} - \tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\dot{\mathbf{u}} - \mathbf{V}\dot{\boldsymbol{\lambda}}) > 0 \tag{7.2}$$

for any $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}} > \mathbf{0}$.

Proof. According to Theorem (I), the solution for any external force set $\dot{\mathbf{F}}$, makes zero the function $Q(\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}})$ equation (3.21); hence the expression (7.1) equals the second order work $\tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}}$ (to within $\frac{1}{2}\delta t^2$).

Note that condition (7.1) is, generally, not necessary, since vectors $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}} > \mathbf{0}$, might make (7.1) negative without being a solution of any actual incremental problem.

(XVII) *The system is non-unstable in Y (i.e. its equilibrium is stable or neutral) if matrix* $\mathbf{M}$, *equation* (3.13), *is copositive.*

Proof. Copositive $\mathbf{M}$ means that $\tilde{\boldsymbol{\zeta}}\mathbf{M}\boldsymbol{\zeta} \geq 0$ (7.2) for any $\boldsymbol{\zeta} \geq \mathbf{0}$. Calculating this quadratic form (7.2) through (3.13), (3.13b), it appears to equal the l.h.s. of (7.1) and hence L, for any $\dot{\mathbf{F}}$, while the sign restriction on $\boldsymbol{\zeta}$ implies precisely the same restriction on $\dot{\boldsymbol{\lambda}}$.

It is worth comparing the uniqueness criterion (6.4) to the stability criterion (7.1). The former is, clearly, more restrictive (the functionals are the same but the admissible class of the former contains that of the latter).

It may happen that (7.1) holds for $\dot{\mathbf{u}}, \dot{\boldsymbol{\lambda}} \geq \mathbf{0}$, whereas the functional (7.1) vanishes for some $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\lambda}}$ not ≥ 0, and, hence, the response to some load increments may be not unique: this means *stable bifurcation*, which therefore may precede an actual loss of stability, as pointed out by Shanley [1].

It is essential to note that Shanley's effect merges here in a more general context than the traditional one: *it may be due either to geometrical* ($\mathbf{K}_G$) *or to physical* ($\mathbf{H}, \mathbf{V} \neq \mathbf{N}$) *instabilizing effects or to a combination of them.*

For $\mathbf{V} = \mathbf{N}$, $\tilde{\mathbf{H}} = \mathbf{H}$ the present conclusion simply corroborates those of Section 6.2; when $\mathbf{H}$ reduces to a nonnegative scalar, statement (XVI) can be given, through the same transformations as for the corresponding uniqueness criterion, a formulation which coincides with the specialization to small strains of one of the results obtained by Hill in [2]. In the elastic range, lack of uniqueness and loss of stability must occur simultaneously at an "eigenstate", when matrix $\mathbf{K}$ ceases to be positive definite and becomes singular.

8. MATRIX DESCRIPTION VERSUS TENSOR FIELD DESCRIPTION

The minimum principle obtained by Hill [2], if specialized to the cases of large displacements but small strains, can be expressed as follows (all quantities defined in natural state and fixed frame, cf. Ref. [30]; summation convention adopted):

$$\text{minimize} \quad \frac{1}{2}\int_V \sigma_{rs}\dot{u}_{h/r}\dot{u}_{h/s}\,\mathrm{d}V + \frac{1}{2}\int_V \dot{\sigma}_{rs}\dot{\varepsilon}_{rs}\,\mathrm{d}V - \int_V \dot{F}_r\dot{u}_r\,\mathrm{d}V - \int_{S_T} \dot{T}_r\dot{u}_r\,\mathrm{d}S \tag{8.1}$$

subject to:

$$\dot{\varepsilon}_{rs} = \tfrac{1}{2}(\dot{u}_{r/s} + \dot{u}_{s/r} + u_{h/r}\dot{u}_{h/s} + u_{h/s}\dot{u}_{h/r}) \text{ in } V \tag{8.2a}$$

$$\dot{u}_r = \dot{\bar{u}}_r \text{ on } S_u \tag{8.2b}$$

where: σ_{rs}, ε_{rs} are the (symmetric, referred to the original state) stress and strain tensors of Kirchhoff and Green–Lagrange, respectively [30]; F_r, T_r = volume and surface forces; V = volume of the body; S_T, S_u = parts of the boundary where surface forces T_r and, respectively, displacements $\bar{u}_r$ are prescribed; · indicates derivative.

The same extremum property† has been re-established in Section 3.3, (3.29), (3.30), in the form (for $\dot{\boldsymbol{\delta}} = \mathbf{0}$):

$$\text{minimize} \quad \tfrac{1}{2}\tilde{\dot{\mathbf{u}}}\mathbf{K}_G\dot{\mathbf{u}} + \tfrac{1}{2}\tilde{\dot{\boldsymbol{\sigma}}}\dot{\boldsymbol{\varepsilon}} - \tilde{\dot{\mathbf{F}}}\dot{\mathbf{u}} \tag{8.3}$$

$$\text{subject to} \quad \dot{\boldsymbol{\varepsilon}} = \hat{\mathbf{B}}\dot{\mathbf{u}}. \tag{8.4}$$

If the two above formulations are compared, clear correspondences can be observed between single parts which have the same mechanical meaning. The correspondence

† Hill assumes smooth yield surfaces and shows that, at the solution, for any compatible infinitesimal variation $\delta\dot{u}_r$, the first variation of the functional (8.1) vanishes. Note that if allowance is made for "corners", as throughout in this paper, the minimum theorems fail to be "variational" in the above sense, as pointed out by Drucker [15].

between the first terms of (8.1) and (8.3) (those due to the presence of large displacements) is particularly interesting. Moreover it represents, together with the correspondence of the equilibrium equations, in a sense the key for formally transferring results from the matrix theory to the continuous theory and vice versa (as the other correspondences, in pairs, of analogous terms and operators are almost selfevident). Thus, e.g. the "general" minimum principle (I) (in Lagrangian description [30]) reads for $\dot{\boldsymbol{\delta}} = \mathbf{0}$, $\mathbf{V} = \mathbf{N}$:

$$\text{minimize} \quad Q(\dot{u}_r, \dot{\lambda}_r) \equiv \int_V \sigma_{rs}\dot{u}_{h/r}\dot{u}_{h/s}\,\mathrm{d}V + \int_V H_{ij}\dot{\lambda}_i\dot{\lambda}_j\,\mathrm{d}V + \int_V \dot{\varepsilon}^e_{rs}S_{rshk}\dot{\varepsilon}^e_{hk}\,\mathrm{d}V - \int_V \dot{F}_r\dot{u}_r\,\mathrm{d}V - \int_{S_T} \dot{T}_r\dot{u}_r\,\mathrm{d}S \tag{8.5}$$

$$\text{subject to} \quad \frac{\partial\varphi_i}{\partial\sigma_{rs}}S_{rshk}\dot{\varepsilon}^e_{hk} - H_{ij}\dot{\lambda}_j \leq 0 \quad \text{in } V \tag{8.6}$$

$$\left.\begin{aligned} &(S_{rshk}\dot{\varepsilon}^e_{hk}u_{m/r} + \sigma_{rs}\dot{u}_{m/r})_{/s} + (S_{mshk}\dot{\varepsilon}^e_{hk})_{/s} + \dot{F}_m = 0 \quad \text{in } V \\ &(S_{rshk}\dot{\varepsilon}^e_{hk}u_{m/r} + \sigma_{rs}\dot{u}_{m/r})n_s + (S_{mshk}\dot{\varepsilon}^e_{hk})n_s = \dot{T}_m \quad \text{on } S_T \ (n_s = \text{unit normal vector}) \end{aligned}\right\} \tag{8.7}$$

$$\dot{\varepsilon}^e_{rs} = \tfrac{1}{2}(\dot{u}_{r/s} + \dot{u}_{s/r} + u_{h/r}\dot{u}_{h/s} + u_{h/s}\dot{u}_{h/r}) - \frac{\partial\varphi_i}{\partial\sigma_{rs}}\dot{\lambda}_i; \qquad \dot{\lambda}_i \geq 0 \text{ in } V,\ \dot{u}_r = \dot{\bar{u}}_r \text{ on } S_u. \tag{8.8}$$

The conformity and equilibrium relations (8.6) and (8.7) correspond to (3.18b) and (3.19) respectively, provided that in the latter pair the expression (3.27a) of $\dot{\varepsilon}^e$ be made explicit.

The above remarks by no means detract from the interest of discussing the results of this paper in the framework of the functional theory. It is worth noting, moreover, that, even for the compatible models adopted here, the convergence of the discrete solution on the continuous solution as the finite element number increases, can be reasonably conjectured, but, outside the elastic range, it has not been rigorously proved so far, to the author's knowledge.

9. ON THE NUMERICAL TECHNIQUES

As it has been emphasized in the Introduction, one of the advantages of the present approach to plastic structural analysis rests on the close connections between theoretical conclusions and numerical procedures which can be used for solving actual engineering problems. Some of these procedures are indicated here below as a first orientation.

9.1 *Incremental problems of general kind*

The linearly constrained optimizations stated in (I) and (IV) concern generally non-convex quadratic functions. So far the only rigorous procedure for carrying them out, is the method of Ritter [31]. This is still little known and only very recently intensively studied, see e.g. Ref. [32].

9.2 *Incremental problems which exhibit positive semidefinite but not symmetric matrices* **M**, (3.13) *and* **M*** (4.5), (4.10a)

Theorems (I) and (IV) reduce these problems to convex quadratic programs which can be solved by means of traditional algorithms (cf. below in 9.3). However it seems more convenient in this class of cases to consider directly either the $\dot{\mathbf{u}}, \dot{\lambda}, \dot{\boldsymbol{\varphi}}$ formulation (Section 3.1) or the $\dot{\lambda}, \dot{\boldsymbol{\varphi}}$ formulation (Section 4.1) as a linear complementarity problem (see Appendix) and to solve it e.g. by means of the algorithms established by Cottle and Dantzig, Cottle ("principal pivoting" methods), Lemke and Graves [33]. The first three methods actually hold for a slightly wider class of matrices; they are expounded with references to the original works, in [34]. All are based on Dantzig's Simplex method and solve the problem in a finite number of pivotal steps or demonstrate the inconsistency.

9.3 *Incremental problems with symmetric, positive semidfinite matrices* $\mathbf{M}$ *and* $\mathbf{M}^*$ [according to the hypothesis (3.21) and (4.14)] can be dealt with by means of the Theorems (II), (III), (V), (VI), which reduce them to convex quadratic programming problems of special or general nature. Several well known algorithms with termination in a finite number of steps, have been proposed and widely applied in different fields. Books in current use e.g. [35, 36] expound them, and sometimes [37] also the relevant computer programs. In cases of convex but not strictly convex objective functions the remarks of Section 6.3 should be kept in mind by applying the dual Theorems (III), (VI).

9.4 *Checking stability, uniqueness of solution and convexity* involves the study of the nature of suitable matrices.

Positive semidefiniteness (definiteness) is synonimous of convexity (strict convexity) of the relevant quadratic form: it is to be tested not only in order to ensure uniqueness or stability according to some conclusions reached in the paper, but also before applying the procedures mentioned in 9.2 and 9.3 to incremental structural problems in advanced stages of a loading process. Among the necessary and sufficient criteria, the most advantageous one seems to be that founded on a sequence of pivot operations on the matrix: it requires no more than $\frac{1}{3}n^3$ multiplications, n being the matrix order [38]. The determinantal criteria are lengthy [39]. Well known, only sufficient criteria can be preferable or of immediate use in many cases. *Copositivity*, postulated e.g. by statement (XVII), is more difficult to check: determinantal tests have been established e.g. in [40] and very recently in [41].

P-matrices, see e.g. statement (X), are characterized in various ways e.g. in [42] and discussed in [43].

Besides a general evaluation of the practical use of the theory developed here, various important questions are to be answered on the basis of future computational experience. Among them we mention here: (a) the relative merit of the $\dot{\mathbf{u}}, \dot{\lambda}, \dot{\boldsymbol{\varphi}}$ formulation and of the $\dot{\lambda}, \dot{\boldsymbol{\varphi}}$ formulation of incremental problems (the latter involves an easier nonlinear stage with less unknowns, but requires the evaluation of matrix $\mathbf{Z}_G$); (b) the better choice, in dubious cases between the direct use of the general method 9.1 and the use of the algorithm quoted in 9.3 and 9.2 preceded by a convexity test.

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APPENDIX

Some notions on complementarity problems and quadratic programming

A.1. The problem of finding vectors $\zeta \in R^p$, $\tilde{\omega} \in R^p$ satisfying

$$\tilde{\omega} = \mathbf{M}\zeta + \mathbf{q} \tag{A1a}$$

$$\omega \geq \mathbf{0}, \qquad \zeta \geq \mathbf{0}, \qquad \tilde{\omega}\zeta = 0 \tag{A1b, c, d}$$

is referred to as a *linear complementarity problem* [44, 34, 45]. R^p is the p-dimensional Euclidean space, $\mathbf{M}$ a given $p \times p$-matrix, $\mathbf{q}$ a given p-vector: all numerical quantities are real.

The nonlinear ortogonality requirement (A1d) is called a *complementarity condition*, since it implies, if associated with the sign constraints, that in each pair of corresponding variables $\omega_i \zeta_i$ one must vanish if the other is positive.

Consider the *quadratic programming problem*:

$$\text{minimize} \quad Q \equiv \tilde{\zeta}\mathbf{M}\zeta + \tilde{\mathbf{q}}\zeta \tag{A2a}$$

$$\text{subject to} \quad \mathbf{M}\zeta + \mathbf{q} \geq \mathbf{0}, \qquad \zeta \geq \mathbf{0}. \tag{A2b, c}$$

Since the objective function $Q(\zeta)$ equals $\tilde{\omega}\zeta$, it can be immediately proved that: if (A1) has a solution, this solution also solves (A2); conversely, if the minimum (optimal value) of problem (A2) is zero, then any optimal solution to (A2) also solves (A1), otherwise (A1) has no solution.

Matrix $\mathbf{M}$ may be not positive semidefinite and, hence, program (A2) not convex: in this case a local minimum does not necessarily represent a global minimum.

A.2. Consider the two sets of relations:

$$\begin{array}{ll|ll} \underline{\mathbf{c}-\tilde{\mathbf{A}}\boldsymbol{\eta}}+\mathbf{D}\boldsymbol{\xi} = \mathbf{x} & & \underline{-\mathbf{b}}+\mathbf{A}\boldsymbol{\xi}+\mathbf{E}\boldsymbol{\eta} = \mathbf{y} & \\ \boldsymbol{\xi} \geq \mathbf{0}, & \mathbf{x} \geq \mathbf{0} & \boldsymbol{\eta} \geq \mathbf{0}, & \mathbf{y} \geq \mathbf{0} \\ \tilde{\mathbf{x}}\boldsymbol{\xi} = 0 & & \tilde{\mathbf{y}}\boldsymbol{\eta} = 0 & \end{array} \tag{A3}$$

where $\mathbf{x}, \boldsymbol{\xi} \in R^n$, $\mathbf{y}, \boldsymbol{\eta} \in R^m$ denote variable vectors: the given entities $\mathbf{D}$, $\mathbf{E}$, $\mathbf{A}$ are $n \times n$, $m \times m$, $m \times n$ matrices and $\mathbf{b}$ and $\mathbf{c}$ m- and n-vectors, respectively. If the underlined terms were assigned, each relation set would reduce to a complementarity problem. On the other hand the whole set (A3) can be put in the form (A1) by means of the identifications:

$$\mathbf{q} \equiv \begin{bmatrix} \mathbf{c} \\ \hdashline -\mathbf{b} \end{bmatrix}, \qquad \mathbf{M} \equiv \begin{bmatrix} \mathbf{D} & -\tilde{\mathbf{A}} \\ \hdashline \mathbf{A} & \mathbf{E} \end{bmatrix}, \qquad \boldsymbol{\zeta} \equiv \begin{bmatrix} \boldsymbol{\xi} \\ \hdashline \boldsymbol{\eta} \end{bmatrix}, \qquad \boldsymbol{\omega} \equiv \begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{y} \end{bmatrix}. \tag{A4a, b, c, d}$$

Assuming $\mathbf{D}$ and $\mathbf{E}$ symmetric nonnegative definite, [hypothesis (A5)], let us take into consideration the (convex) quadratic program:

$$\left.\begin{array}{ll} \text{minimize} & Q_{\mathrm{I}} \equiv \tilde{\mathbf{c}}\boldsymbol{\xi}+\tfrac{1}{2}\tilde{\boldsymbol{\xi}}\mathbf{D}\boldsymbol{\xi}+\tfrac{1}{2}\tilde{\boldsymbol{\eta}}\mathbf{E}\boldsymbol{\eta} \\ \text{subject to} & -\mathbf{b}+\mathbf{A}\boldsymbol{\xi}+\mathbf{E}\boldsymbol{\eta} \geq \mathbf{0}, \qquad \boldsymbol{\xi} \geq \mathbf{0} \end{array}\right\}. \tag{A6}$$

All vectors (points) which comply with the constraints of a programming problem are said to be *feasible* and to form the feasible domain in their space. The *Kuhn–Tucker theorem* [46, 35], interpreted in geometric intuitive terms, states that a feasible point is a solution of a convex program, if, and only if, the gradient of the objective function in the point is a nonpositive linear combination of the outward-directed normals of those support hyperplanes of the feasible domain, which contain that point, if any of such planes exist. By translating this theorem into analytical terms, cf. [35, 36], (A3) are found to represent the above optimal conditions of problem (A6) and, hence, to be entirely equivalent to it, and, via (A4), to a linear complementarity problem (A1). When hypothesis (A5) fails to apply, conditions (A3) are necessary but *not sufficient* for an optimum.

A.3. For $\mathbf{E} = \mathbf{0}$, (A6) becomes the traditional formulation of a general quadratic program in the variable $\boldsymbol{\xi}$. By adding the $\mathbf{E}$-terms, under hypothesis (A5), Cottle developed the *symmetric duality theory* [47, 34] summarized below. The programming problem

$$\begin{array}{ll} \text{maximize} & Q_{\mathrm{II}} \equiv \tilde{\mathbf{b}}\boldsymbol{\eta}-\tfrac{1}{2}\boldsymbol{\xi}\mathbf{D}\tilde{\boldsymbol{\xi}}-\tfrac{1}{2}\tilde{\boldsymbol{\eta}}\mathbf{E}\boldsymbol{\eta} \\ \text{subject to} & \mathbf{c}+\mathbf{D}\boldsymbol{\xi}-\tilde{\mathbf{A}}\boldsymbol{\eta} \geq \mathbf{0}, \qquad \boldsymbol{\eta} \geq \mathbf{0} \end{array} \tag{A7}$$

is the dual of (A6), in the sense that, among others, the following joint properties can be proved [48, 47]:

(a) if either problem is solvable, both are solvable and the extremal values are equal;

(b) if $(\boldsymbol{\xi}_0, \boldsymbol{\eta}_0)$ is a solution of (A6), there exists a vector $\bar{\boldsymbol{\eta}}_0$ such that $(\boldsymbol{\xi}_0, \bar{\boldsymbol{\eta}}_0)$ solves both (A6) and (A7); an analogous statement holds for solutions of (A7);

(c) if the feasible domain is empty in one problem but not in the other, then the objective of the latter on the feasible domain is unbounded in the direction of optimization.

It is easy to see that:

(i) if the dual problem (A7) is written as a minimization problem, its dual is exactly the primal (A6) written as a maximization problem (*symmetry* property) [47];

(ii) if (A2) is interpreted as a special case of (A6) and dualized into (A7), it appears to be *selfdual* in the Dorn sense [49], i.e. coincident with its dual;

(iii) if (A4) are substituted into (A2), this becomes the *composite program* of (A6) and (A7), since it turns out to consist of minimizing the difference of the primal and dual objective functions over the intersection of the two feasible domains; then any optimal solution of (A2) solves (A6) and (A7) as well [47];

(iv) when hypothesis (A5) holds, matrix $\mathbf{M}$, given by equation (A4b), is positive semidefinite as a consequence (but symmetric if, and only if, $\mathbf{A} = \mathbf{0}$). Then the following theorem can be proved [50, 44]: if in (A1) the linear relations are consistent, the whole problem (A1) (including the complementarity condition) is solvable; therefore, if the feasible domain of (A2) is not empty, Q attains the minimum zero over it.

A.4. The entire set of solutions (optimal vectors) of a convex quadratic programming problem is the intersection of the feasible domain with the linear manifold obtained according to the following rule: add to any optimal vector all vectors which make zero the quadratic form of the objective function and, simultaneously, are ortogonal to the constant vector of the linear term of the objective function. This assertion is proved in most standard books on nonlinear programming, e.g. in [35, 36].

A.5. On the solvability and the solutions of linear complementarity problems (A1) we use in the text the following theorems:

(i) a unique solution exists for any $\mathbf{q} \in R^p$ if and only if $\mathbf{M}$ is a matrix of class P (i.e. has all principal minors positive) [51];

(ii) the number of solutions is finite for all $\mathbf{q} \in R^p$, if and only if all the principal minors of $\mathbf{M}$ are nonzero [52].

(iii) if $\mathbf{M}$ is positive semidefinite and the constraints (A2b, c) are consistent, then problem (A1) is solvable; if $\mathbf{M}$ is positive definite, the constraints (A2, b c) are always consistent [cf. (iv) in A.3 and (i) above].

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Абстракт—Предполагаются упруго-пластические конститутивные законы с поправками для следующих вопросов: отступление от нормальности, угловые точки дающие возможность приёма разных видов схем пластического течения и смягчение материала. Сплошная среда заменяется моделями конечного элемента с полной совестимостью повсюду. Уравнения равновесия касаются деформированного состояния, но даже деформации рассматрываются малые. Исследуется структурное поведение по отношению скоростей нагрузок и дислокаций на пример термические деформации. Получаются следующие результаты:

а) шесть свойств экстремума решений, которые сводят задачу комечмых размосмей к задаче квадратического програмимнования;

б) методы для определения верхнего и нижнего пределов для локальной мгновеенной податливости, имея в виду одинарную нагрузку или дислокацию;

в) критерия для единственности решения разностной задачи и для всей устойчивости системы.